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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

. . . *ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum.* LEIBNIZ

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On the Differentiability of Functions of Several Variables

JAMES SERRIN

Let $u=u(x)=u(x_1, \ldots, x_n)$ be a real-valued locally summable function defined on an open region R of the n-dimensional number space $\mathcal{R}^n$. It is our purpose here to investigate the differentiability of u when the gradient of the distribution associated with u is a measure. By this we mean that there exists a Borel measure α, finite-valued on compact subsets of R, such that the formula

$$-\int u\,\varphi_x\,dx=\int \varphi\,d\alpha \tag{1}$$

holds for any continuously differentiable function φ with compact support in R. Here φ_x denotes the vector $\{\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_n\}$, and the measure α is of course vector-valued and unrestricted as to sign. For convenience, a function satisfying the preceding condition will be called *weakly differentiable*.

In order to discuss the differentiability properties of a summable function, whose values are in the nature of things somewhat arbitrary on a set of measure zero, we make the following definition: A function $u(x)$ will be called *essentially differentiable* if there exists an equivalent function $v(x)$, defined on R, which has ordinary partial derivatives $\{\partial v/\partial x_1, \ldots, \partial v/\partial x_n\}$ almost everywhere.

Suppose now that u is essentially differentiable. If v_1 and v_2 are two functions, each one equivalent to u and having partial derivatives almost everywhere, then obviously the partial derivatives of v_1 and v_2 agree almost everywhere. We may therefore define the derivative of u with respect to x_k, $1\leqq k\leqq n$, to be the *class of functions equivalent to* $\partial v/\partial x_k$, where v is any function equivalent to u and having partial derivatives almost everywhere. This derivative will be denoted by $\partial u/\partial x_k$. Although $\partial u/\partial x_k$ is an equivalence class, one naturally thinks of it as a function in the same way as elements of L_p are thought of as functions. Finally, if u is essentially differentiable, we set

$$u_x=\{\partial u/\partial x_1, \ldots, \partial u/\partial x_n\}.$$

The main result of the paper is now the following

Theorem 1. *If u is weakly differentiable, then u is essentially differentiable. Moreover $u_x=\alpha'$, where α' is the general derivative of the set function α.*

The proof of the first statement in Theorem 1 occupies Sections 2 and 3 of the paper, Section 1 being devoted to a simple but interesting preparatory lemma. At the end of Section 3 we digress from the proof of Theorem 1 to obtain an immediate demonstration of a known theorem for the case when u is continuous. The remaining part of Theorem 1 is then proved in Sections 4 and 5, following which we briefly discuss the relation between weak differentiability and bounded variation. The methods of the paper have their partial genesis in the work of Goffman, Krickeberg, and de Vito, and our main theorem complements the results of these authors[1]. It may be worth adding that the only background necessary for the present work is a knowledge of the elements of the theory of integration, and the Lebesgue decomposition theorem.

Remarks. There are several immediate consequences of Theorem 1 which may be conveniently noticed at this point.

1. By use of a well known fact about distributions [*11*, pp. 37—38], it follows from the first part of Theorem 1 that if T is a distribution whose gradient is a measure, then T is defined by a function having partial derivatives almost everywhere.

2. An essentially differentiable function obviously has approximate partial derivatives almost everywhere. Therefore, by Theorem 1 and a result of Stepanoff [*10*, p. 300], a weakly differentiable function u has an approximate differential almost everywhere. On the other hand, neither u nor any equivalent function need have an ordinary differential at any point. The situation with respect to the existence of a differential is discussed more fully in the concluding sections, 8 and 9, of the paper.

3. Finally, by virtue of the formula $u_x = \alpha'$, one sees that u_x is locally summable and transforms as a Cartesian vector.

1. A preparatory lemma

The results of this paper are ultimately based on the well known theorem of Lebesgue which states that any monotone function of a single real variable has a finite derivative almost everywhere. The following lemma generalizes this result to functions of several variables.

Lemma 1. *Let $f(x)$ be a measurable function defined on an open interval I in $\mathscr{R}^n$. Suppose that $f(x)$ is monotone on almost all lines parallel to the x_k axis. Then the partial derivative $\partial f/\partial x_k$ exists and is finite almost everywhere in I.*

Before proving this result, some comments are in order. Since Lebesgue's theorem for one variable also applies to functions of bounded variation, one might conjecture that the hypothesis of Lemma 1 could be weakened from "monotone on almost all lines" to simply "bounded variation on almost all lines". This change, however, would destroy the lemma: There exist measurable functions of uniformly bounded variation on *all* lines parallel to the x_1 axis, say, which do *not* possess a partial derivative $\partial f/\partial x_1$ almost everywhere in I. As a matter of interest, such an example is constructed in Section 7.

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[1] Although our results might seem to follow from Theorems XII and XIV of [*3*], this is not actually the case. In particular, in [*3*] each component of u_x is treated on the basis of a different function $v(x)$, and, more important, the derivatives $\partial u/\partial x_k$ which appear in [*3*] are defined by the special operation $\lim_{n\to\infty} \Delta u/\Delta x_k$, where $\Delta x_k = 1/n$; *cf.* footnote 6 of [*3*].

Proof of Lemma 1. It is clear from the hypothesis that $\partial f/\partial x_k$ exists and is finite almost everywhere on almost all lines parallel to the x_k axis. It does not follow from this, of course, that $\partial f/\partial x_k$ exists almost everywhere in I. We must therefore resort to further argument.

For convenience, the set of variables $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ will be denoted by y, and the set $(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)$ by (t, y). Now for $x=(x_k, y)\in I$ we let

$$E^+ f = E^+ f(x) = \limsup_{m\to\infty} \frac{f(x_k+1/m, y)-f(x_k, y)}{1/m} \qquad (m = \text{integer})$$

be an upper right "sequential derivate" of f. It is clear that $E^+ f$ is measurable, and, as one easily deduces, finite almost everywhere in I. The same result holds, of course, for the lower right derivate $E_+ f$, and the left derivates $E^- f$ and $E_- f$. Now by Fubini's theorem

$$\int_I |E^+ f - E_+ f|\, dx = \int \left[\int |E^+ f - E_+ f|\, dx_k\right] dy = 0,$$

since the inner integral vanishes for almost all y. It follows at once that both limits

$$\lim_{m\to\infty} \frac{f(x_k+1/m, y)-f(x_k, y)}{1/m}, \qquad \lim_{m\to\infty} \frac{f(x_k-1/m, y)-f(x_k, y)}{-1/m}$$

exist and are equal almost everywhere in I. Let $E(x)$ denote the common limit where it exists; otherwise put $E(x)=\infty$. $E(x)$ is obviously finite-valued almost everywhere in I.

Now let T denote the set of points $x\in I$ such that $E(x)$ is finite and $f(x)$ monotone in x_k. If $x\in T$ and $1/m+1 \leqq \Delta x_k \leqq 1/m$, then

$$\frac{f(x_k+1/m+1, y)-f(x_k, y)}{1/m} \leqq \frac{\Delta f}{\Delta x_k} \leqq \frac{f(x_k+1/m, y)-f(x_k, y)}{1/m+1},$$

provided f is monotone *increasing* at x (when f is *decreasing* the inequalities must be reversed). Since both the right and left sides of the above inequality tend to $E(x)$ as $m\to\infty$, it follows that $\partial f/\partial x_k$ exists and is finite at each point of T. But T includes almost all points of I, and the proof is thereby completed.

2. Proof of Theorem 1. First part

Consider any fixed open interval I in $\mathscr{R}^n$ whose closure is contained in R, and let k be a fixed integer, $1\leqq k\leqq n$. In order to prove the first statement of Theorem 1, it will be enough to show that there is a function $v(x)$, equivalent to u *and* independent of k, such that $\partial v/\partial x_k$ exists almost everywhere in I. The first step in this process is to obtain an estimate of the variation of u along lines parallel to the x_k axis.

Let us associate with each positive number h a corresponding function $K(\xi)\equiv K(\xi, h)$ with the properties:

i) K is non-negative and smooth for all values of $\xi=(\xi_1, \ldots, \xi_n)$.

ii) $K\leqq \text{Const.}/h^n$ for $|\xi|<h$ and $K\equiv 0$ for $|\xi|\geqq h$.

iii) $\int K(\xi)\, d\xi = 1$.

The ξ integration in this and subsequent formulas is assumed to be carried out over the entire n-dimensional number space. For $x \in I$ (and for h sufficiently small), we define the function

$$u_h = u_h(x) = \int K(\xi - x)\, u(\xi)\, d\xi. \tag{2}$$

In view of the properties of K, it is easy to prove that $\lim u_h$ exists and equals u almost everywhere in I.[2] We set $w = \lim u_h$ wherever the limit exists, and otherwise leave the function w undefined. Clearly $w = u$ almost everywhere.

It will be convenient at this stage to introduce some simple notation. We consider the interval I to be the Cartesian product $J \times J'$ of an interval J of the variable x_k and an interval J' of the complementary variable $y = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. Correspondingly, the set of variables $(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)$ will be denoted by (t, y) as in the preceding section. Finally, for any set B in $\mathscr{R}^m$ we let B_h denote the set of points in B whose distance from $\mathscr{R}^m - B$ is greater than h.

Now by differentiation of (2) and use of (1), there results

$$\frac{\partial u_h}{\partial x_k} = -\int \frac{\partial K}{\partial \xi_k}(\xi - x)\, u(\xi)\, d\xi = \int K(\xi - x)\, d\alpha_k, \tag{3}$$

where α_k is the k^{th} component of the measure α. An easy application of Fubini's theorem then yields

$$\int\limits_{I_h} \left| \frac{\partial u_h}{\partial x_k} \right| dx \leqq \int\limits_{I} d|\alpha_k|.$$

For $y \in J'$ let us define

$$V(y) = \liminf \int\limits_{J_h} \left| \frac{\partial u_h}{\partial x_k}(t, y) \right| dt, \qquad h \in \{1/m\}.$$

Clearly $V(y)$ is measurable on J', whence by Fatou's lemma

$$\int\limits_{J'} V(y)\, dy \leqq \int\limits_{I} d|\alpha_k|$$

(integrate first over a compact subset K of J', and then let $K \to J'$). Now let $\varphi(x)$ be a real-valued function defined a.e. on I. For $y \in J'$, let $\operatorname{Var} \varphi$ be the function defined by

$$[\operatorname{Var} \varphi](y) = \underset{t \in J}{\text{Total variation}}\, \varphi(t, y)$$

with the convention that if $\varphi(x)$ is undefined at a certain set of points, then these points are simply to be disregarded in calculating the total variation. In view of the definition of $V(y)$, it is clear that the variation of the function $w = \lim u_h$ satisfies $\operatorname{Var} w \leqq V(y)$ for $y \in J'$. This completes the first step in the proof.

---

[2] Indeed, we have

$$|u_h(x) - u(x)| = \left|\int K(\xi - x)\,[u(\xi) - u(x)]\, d\xi\right| \leqq \frac{\text{Const.}}{h^n} \int\limits_{|\xi - x| \leqq h} |u(\xi) - u(x)|\, d\xi,$$

and the last expression tends to zero at each point of the Lebesgue set of u.

3. Continuation

The function $w(x)$ constructed in the preceding section has the property

$$\operatorname{Var} w \leqq V(y), \qquad \int_{J'} V(y)\,dy < \infty. \tag{4}$$

Moreover, w is equal to u almost everywhere in I, hence is summable. Such functions allow a canonical decomposition into the difference of two monotone functions, according to the following lemma.

Lemma 2. *Let $w(x)$ be a summable function defined a.e. on an open interval I, and there satisfying with respect to some variable x_k the condition* (4). *Then there exist two summable functions w^+ and w^- on I, each monotone increasing in the variable x_k, such that almost everywhere in I*

$$w = w^+ - w^-. \tag{5}$$

Remark. A representation of the type (5) is not always possible if the equality is to hold *everywhere* in I. Indeed, if it were possible, then by Lemma 1 any function satisfying the hypotheses of Lemma 2 would have a partial derivative $\partial w/\partial x_k$ almost everywhere in I. But according to the example of Section 7, this is not the case.

Proof of Lemma 2. We shall first construct a function $\omega(x)$, equivalent to $w(x)$ but having "fewer" discontinuities. Let S denote the set of points $y \in J'$ such that

i) $w(x_k, y)$ is defined for almost all $x_k \in J$, and
ii) $V(y) < \infty$.

Obviously S includes almost all points of J'. For each $x = (x_k, y) \in I$ let us then define

$$\omega(x) = \begin{cases} w(x_k - 0, y) & y \in S, \\ 0 & y \notin S. \end{cases}$$

$\omega(x)$ is measurable, since it is the limit almost everywhere of the sequence of measurable functions $\{w(x_k - 1/m, y)\}$. Now on almost every line $y = y_0$ the functions ω and w differ at most at a set of linear measure zero. From this fact, and since ω and w are both measurable, it follows easily that $\omega = w$ almost everywhere in I. The argument above shows not only that $\omega(x)$ is measurable, but also, by a slight variant, that each function $\omega(x_k, y)$ is measurable in y for fixed x_k.

Now let $x_k = a$ be a point such that

$$\int_{J'} \omega(a, y)\,dy < \infty$$

(such a point exist, by Fubini's theorem). Also, for fixed y, let $\tau(x_k, y)$ be the indefinite total variation of the function $\omega(x_k, y)$, normalized so that $\tau(a, y) = 0$. We assert that the function $\tau(x) = \tau(x_k, y)$ is measurable on I.

To prove this, denote by P_m the partition of the interval J by m equally spaced points $t_1 < t_2 < \cdots < t_m$. The *approximate variation* of $\omega(x_k, y)$ corresponding to the partition P_m is then

$$\tau_m(x_k, y) = \sum |\omega(t_i, y) - \omega(t_{i-1}, y)| \cdot \operatorname{sign}(x_k - a),$$

where the sum is extended over all partition points t_i which lie in the interval between a and x_k. Since ω is continuous from the left, it is easily seen that

$$\tau(x_k, y) = \lim_{m\to\infty} \tau_m(x_k, y).$$

Obviously $\tau_m(x_k, y)$ is a step function in x_k for y fixed. More specifically, for x_k between any two consecutive points of P_m, we have

$$\tau_m(x_k, y) = \text{finite sum of measurable functions of } y.$$

Thus $\tau_m(x) = \tau_m(x_k, y)$ is a measurable function, and since $\tau(x) = \lim \tau_m(x)$, so also is $\tau(x)$ measurable.

We can now define

$$w^+(x) = \tfrac{1}{2}[\tau(x) + \omega(x)], \qquad w^-(x) = \tfrac{1}{2}[\tau(x) - \omega(x)].$$

Clearly w^+ and w^- are measurable, monotone in the variable x_k, and $w^+ - w^- = w$ almost everywhere in I. Furthermore

$$\begin{aligned} |w^+| + |w^-| &\leqq \{\operatorname{Var} w^+ + \tfrac{1}{2}|\omega(a, y)|\} + \{\operatorname{Var} w^- + \tfrac{1}{2}|\omega(a, y)|\} \\ &= \operatorname{Var}\omega + |\omega(a, y)| \leqq V(y) + |\omega(a, y)|, \end{aligned}$$

so that w^+ and w^- are summable. This completes the proof of Lemma 2.

We now return to the proof of the first statement of Theorem 1. Consider the functions $(w^+)_h$ and $(w^-)_h$ defined as in equation (2). Let S' denote the set of points $y_0 \in J'$ such that $\lim (w^+)_h = w^+$ and $\lim (w^-)_h = w^-$ almost everywhere on the line $y = y_0$. Clearly S' includes almost all points of J'.

Let us fix attention upon any point $x = (x_k, y)$ for which $y \in S'$ We assert that if $\lim (w^+)_h$ does not exist at x, then $\lim (w^-)_h$ does exist at this point. Indeed, since $(w^+)_h$ and w^+ are monotone, if $\lim (w^+)_h$ does *not* exist at x then the relation

$$b_1 \leqq \liminf (w^+)_h < \limsup (w^+)_h \leqq b_2$$

holds at x, where b_1 and b_2 are the left and right hand limits of w^+ at x. Thus w^+ is discontinuous at x. But then w^- must be continuous there (recall that w^+ and w^- are the positive and negative variations of a function *continuous* from the left), and hence $\lim (w^-)_h = w^-$.

We are now in a position to define the function $v(x)$ promised at the beginning of the proof. In particular, put

$$v(x) \equiv \limsup u_h(x).$$

Clearly v is equivalent to u and independent of k To show that v is differentiable, we introduce functions $w_1(x)$ and $w_2(x)$ by means of

$$w_1(x) = \begin{cases} \limsup (w^+)_h & y \in S' \\ v(x) & y \notin S', \end{cases} \qquad w_2(x) = \begin{cases} \liminf (w^-)_h & y \in S' \\ 0 & y \notin S'. \end{cases}$$

w_1 and w_2 are equivalent to w^+ and w^-, respectively, and are monotone on almost all lines parallel to the x_k axis. Furthermore

$$v(x) \equiv w_1(x) - w_2(x), \qquad x \in I,$$

as follows from the identity $u_h=(w^+-w^-)_h=(w^+)_h-(w^-)_h$ and the assertion of the preceding paragraph. By Lemma 1, w_1 and w_2 are differentiable with respect to x_k at almost all points of I. The same is therefore true of v, and the first part of the proof is completed.

Digression. The preceding proof when applied to a continuous function u leads to the following important result.

Theorem 2. *If u is continuous and weakly differentiable, then u has partial derivatives almost everywhere.*

Proof. If u is continuous, then $\lim u_h=u$ uniformly on any closed subregion of R. Thus in the preceding construction $v\equiv u$, and the partial derivative $\partial u/\partial x_k$ exists almost everywhere. Q.E.D.

4. Proof of Theorem 1. Second part

Let $v(x)$ be any function equivalent to u which has partial derivatives almost everywhere, and let k be a fixed integer, $1\leqq k\leqq n$. We must show that

$$\partial v/\partial x_k=\alpha_k' \quad \text{almost everywhere in } R. \tag{6}$$

In the proof, which will be somewhat indirect, we shall maintain the notational conventions adopted in Section 2; furthermore, for simplicity in printing, we shall use the tensor notation $\varphi_{,k}$ for a partial derivative $\partial\varphi/\partial x_k$.

From here on, let $F(p)$ be a non-negative convex function of the single real variable p, with the property that

$$F(p+q)\leqq F(p)+|q|, \tag{7}$$

equivalent to the condition $F(p)\leqq|p|+\text{const}$. For any interval I whose closure is contained in R, we assert that

$$\int_I F(v_{,k})\,dx\leqq\liminf\int_{I_h} F(u_{h,k})\,dx. \tag{8}$$

To begin with, for any continuously differentiable function $\varphi(x)$ one has the relation

$$F(\Delta\varphi/\Delta x_k)=F\left(\frac{1}{\delta}\int_{x_k}^{x_k+\delta}\varphi_{,k}(t,y)\,dt\right)\leqq\frac{1}{\delta}\int_{x_k}^{x_k+\delta}F(\varphi_{,k})\,dt,$$

by Jensen's inequality; we have put $\delta=\Delta x_k$ for convenience in printing. Integrating this inequality along the interval J_δ, where $I=J\times J'$ as in the preceding sections, yields

$$\int_{J_\delta}F(\Delta\varphi/\Delta x_k)\,dx_k\leqq\frac{1}{\delta}\int_{J_\delta}\int_{x_k}^{x_k+\delta}F(\varphi_{,k})\,dt\,dx_k$$

$$\leqq\frac{1}{\delta}\int_J\int_{t-\delta}^{t}F(\varphi_{,k})\,dx_k\,dt=\int_J F(\varphi_{,k})\,dt.$$

Integrating again over J_δ', one then obtains

$$\int_{I_\delta}F(\Delta\varphi/\Delta x_k)\,dx\leqq\int_I F(\varphi_{,k})\,dx. \tag{9}$$

Setting $\varphi = u_h$ in (9), replacing I by I_h, and then letting $h \to 0$, we deduce from Fatou's lemma that

$$\int_{I_\delta} F(\Delta u/\Delta x_k)\, dx \leqq \liminf \int_{I_h} F(u_{h,k})\, dx.$$

Now let u be replaced by v in the left-hand side of this inequality, and allow δ to tend to zero. This proves (8).

5. Completion of the proof

Here we shall show in contrast to (8) that

$$\int_{I_h} F(u_{h,k})\, dx \leqq \int_I F(\alpha_k')\, dx + \Theta(I), \tag{10}$$

where Θ is a singular measure.

The measure $\alpha(E)$ can be expressed by means of the Lebesgue decomposition theorem in the form

$$\alpha(E) = \int_E \alpha'(x)\, dx + \vartheta(E),$$

where $\vartheta(E)$ is a singular measure. Accordingly, equation (3) can be written

$$u_{h,k} = \int K(\xi - x)\, \alpha_k'(\xi)\, d\xi + \int K(\xi - x)\, d\vartheta_k.$$

Therefore by virtue of (7) and Jensen's inequality

$$\begin{aligned} F(u_{h,k}) &\leqq F\left(\int K(\xi - x)\, \alpha_k'(\xi)\, d\xi\right) + \left|\int K(\xi - x)\, d\vartheta_k\right|, \\ &\leqq \int K(\xi - x)\, F(\alpha_k')\, d\xi + \int K(\xi - x)\, d|\vartheta_k|. \end{aligned}$$

Integrating over I_h and applying Fubini's theorem yields (10), with $\Theta = |\vartheta_k|$.

We can now prove (6) and thus complete the proof of Theorem 1. By combining (8) and (10) one obtains

$$\int_I F(v_{,k})\, dx \leqq \int_I F(\alpha_k')\, dx + \Theta(I).$$

Dividing each side by $|I|$ and letting I tend regularly to x gives, on account of Lebesgue's differentiation theorem [*10*, pp. 118—119],

$$F(v_{,k}) \leqq F(\alpha_k') \quad \text{almost everywhere in } R. \tag{11}$$

Now choose

$$F(p) = F_1(p) = \begin{cases} e^p & p < 0, \\ p + 1 & p \geqq 0, \end{cases}$$

which certainly satisfies the imposed conditions. Since $F_1(p)$ is monotone increasing, it follows from (11) that $v_{,k} \leqq \alpha_k'$ almost everywhere. Similarly, setting $F(p) = F_2(p) = F_1(-p)$, we find in the same way that $v_{,k} \geqq \alpha_k'$ almost everywhere. Combining the last two inequalities completes the proof of Theorem 1.

6. Bounded variation and weak differentiability

A function $u(x)$ will be called of *bounded variation in the sense of Cesari* if there is an equivalent function $w(x)$ defined a.e. such that, for every open interval I in the interior of the domain of definition of u, and for each variable x_k,

$1 \leqq k \leqq n$, we have $$\operatorname{Var} w \leqq V(y), \quad y \in J',$$

for some function $V(y)$ which is summable on the interval J'. The function $V(y)$ may of course be different for different directions x_k.

Theorem 3. *A function u is weakly differentiable if and only if it is of bounded variation in the sense of Cesari*[3].

Proof. The "only if" part of this theorem has already been proved in Section 2. Suppose then that u is of bounded variation in the sense of Cesari. By Theorem 5.4 of reference [6], the function u is locally summable.

Now let $C(I)$ denote the linear space whose members are real-valued continuous functions on I, with compact support in I. Similarly, let $C'(I)$ denote the space of continuously differentiable functions with compact support in I. For $\varphi \in C'(I)$ and for any fixed k, $1 \leqq k \leqq n$, set

$$A(\varphi) - \int_I u \, \varphi_{,k} \, dx.$$

Evidently $A(\varphi)$ is a linear operator on $C'(I)$. Moreover, A is continuous in the topology of $C(I)$. To see this, let us observe first that $\operatorname{Var} w < \infty$ for almost all $y \in J'$. Thus for almost all y there exists a corresponding measure β_y on J such that

$$w(x_k, y) = \int^{x_k} d\beta_y, \quad \text{and} \quad \int_J d|\beta_y| \leqq V(y).$$

It follows that

$$A(\varphi) = \int_{J'} \left[\int_J u \, \varphi_{,k} \, dx_k \right] dy = \int_{J'} \left[\int_J \varphi \, d\beta_y \right] dy,$$

and so

$$|A(\varphi)| \leqq \operatorname*{Max}_I |\varphi| \cdot \int_{J'} V(y) \, dy,$$

proving our assertion. Since $A(\varphi)$ is continuous in $C(I)$, it may be extended to a continuous linear functional on $C(I)$. But then by the Riesz theorem there exists a Borel measure α_k such that

$$A(\varphi) = -\int \varphi \, d\alpha_k.$$

This equality, however, is precisely the condition that u should be weakly differentiable on I. Since I was any interval in R, it follows easily that u is weakly differentiable in R.

Theorem 4. *If u is of bounded variation in the sense of Cesari, then for any interval $I \subset R$ and for any k, $1 \leqq k \leqq n$, we have*

$$\int_{J'} V(y) \, dy \geqq \int_I |\partial u / \partial x_k| \, dx.$$

Proof. By the preceding theorem, u is weakly differentiable in I, and the measure α_k even satisfies

$$\int_I d|\alpha_k| \leqq \int_{J'} V(y) \, dy.$$

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[3] This result is due to Krickeberg; the present proof, which in part parallels that of Krickeberg, is given mainly for the sake of completeness.

On the other hand, it is clear that

$$\int_I d|\alpha_k| \geqq \int_I |\alpha_k'|\, dx = \int_I |u_{,k}|\, dx,$$

and the theorem is proved.

7. A counterexample

For the purpose of this section we shall suppose that I is the unit square, $0<x<1$, $0<y<1$, in the plane of the two real variables x, y. As indicated in Section 1, we intend to construct a function $g(x, y)$ on I with the properties

i) $g=0$ almost everywhere in I,

ii) $\operatorname{Var} g = 1$ on each line $y=$ constant,

and nevertheless such that

iii) $\partial g/\partial x$ does not exist at a set of points whose outer measure is one.

The construction is based on the fact that there exists a set O in I which has outer measure one, and yet has only two points in common with each straight line parallel to the x-axis (see [*12*]).

With each point $Q\in O$ let us associate a sequence of points Q_m which lie on the same horizontal line as Q, have rational abscissas x_m, and tend to Q as $m\to\infty$. By omitting certain of the points Q_m and then renumbering the sequence, we can suppose furthermore that

$$0<\sum |Q-Q_m|<1 \tag{12}$$

and that the sets $\{Q_m\}$ corresponding to the two different points of O on any horizontal line are disjoint. The function $g(x, y)$ can now be defined. First, at any point $(x, y)=Q_m$ associated with a point $Q\in O$, we set

$$g(x, y) = |Q-Q_m|/4\Sigma$$

where $\Sigma=\Sigma(Q)$ denotes the sum of the series (12). Then, at all other points, we put $g(x, y)=0$.

The resulting function is zero so long as x is irrational, hence property i) is assured. Property ii) is likewise immediately clear. Finally, at any point $Q\in O$ we have

$$\liminf \left|\frac{\Delta g}{\Delta x}\right| = 0, \qquad \limsup \left|\frac{\Delta g}{\Delta x}\right| = 1/4\Sigma.$$

Thus $\partial g/\partial x$ does not exist at any point of O. Q.E.D.

Remark. Although Lemma 1 is thus shown to be incorrect under the hypothesis of bounded variation, a slightly modified version remains true, namely

Lemma 1'. *Let $f(x)$ be a measurable function defined on an open interval I in $\mathscr{R}^n$. Suppose that f is of bounded variation on almost all lines parallel to the x_k axis. Then there exists an equivalent function $h(x)$ which has finite partial derivatives almost everywhere in I.*

The proof is based on the fact that $f(x)$ can be decomposed into two monotone functions as in the proof of Lemma 2. We omit the details.

8. The existence of a differential

We have already remarked in the introduction that a weakly differentiable function need not have an ordinary differential at any point. It is the purpose of this section to investigate the precise situation with respect to the existence of a differential. This can best be done by introducing the concept of strong differentiability. A locally summable function $u(x)$ will be said to be *strongly differentiable* if there exists a vector-valued function $\psi(x)$ such that the formula

$$-\int u\,\varphi_x\,dx = \int \varphi\,\psi\,dx \tag{13}$$

holds for every continuously differential function φ with compact support in R. Equivalently, u is strongly differentiable if it is weakly differentiable and the associated measure is absolutely continuous. By Theorem 1 a strongly differentiable function is essentially differentiable and has the derivative $u_x = \psi$. The following result now holds.

Theorem 5. *Suppose that $u(x)$ is strongly differentiable and that u_x belongs to L_p for some $p > n$. Then u is continuous and has a differential almost everywhere.*

The conclusion of the theorem refers of course only to a particular member of the equivalence class of u.

For the case of two independent variables, Theorem 5 is essentially due to Cesari [*2*]. The proof we give is a modification and simplification of his. Theorem 5 is best possible: If u is strongly differentiable and u_x belongs to L_n, then u need not be continuous, and, even if it is continuous, it need not possess a differential at any point. Examples illustrating this behavior are given in the following section.

Proof of Theorem 5. It is well known that if u is strongly differentiable and u_x is in L_p, then there exists a sequence of continuously differentiable functions $u_m(x)$ such that

$$\int_S |u_m - u|\,dx \to 0, \qquad \int_S |u_{m\,x} - u_x|^p\,dx \to 0, \tag{14}$$

where S is any compact subregion of R.[4] Now let x and y be two points of R, and let G denote the sphere whose diameter is the line segment joining x and y. If G has compact support in R, then

$$|u_m(x) - u_m(y)| \leqq \text{Const.}\,|x-y|^{1-n/p}\left(\int_G |u_{m\,x}|^p\,dx\right)^{1/p} \qquad (p > n), \tag{15}$$

where the constant depends only on n and p. A simple proof of (15) may be found in reference [*8*], pp. 523–524.

From (14) and (15) one easily infers that the sequence $\{u_m(x)\}$ is uniformly bounded and equicontinuous in any compact subregion of R. Consequently there exists a subsequence which converges to a continuous function $v(x)$ in R. Obviously $v = u$ almost everywhere, and the first part of Theorem 5 is thereby proved.

We may assume from here on that u is a continuous function, and that $u_m \to u$ uniformly on any compact subregion of R. Letting $m \to \infty$ in inequality (15) then yields

$$|u(x) - u(y)| \leqq \text{Const.}\,|x-y|^{1-n/p}\left(\int_G |u_x|^p\,dx\right)^{1/p}. \tag{16}$$

(Hence $u(x)$ is Hölder continuous in R, a result due to Morrey.)

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[4] See reference [*4*]. We can in fact take $u_m = u_h$ with $h = 1/m$.

Now let E denote the Lebesgue set of u_x relative to the class L_p.[5] We shall show that u has a differential at each point x_0 in E, and this will prove the theorem. Indeed, consider the function

$$w(x) = u(x) - A \cdot x,$$

where A denotes the constant vector $u_x(x_0)$. Evidently $w(x)$ is strongly differentiable and $w_x = u_x - A$. Therefore, as one sees by comparing with (16),

$$|w(x) - w(x_0)| \leqq \text{Const.}\, |x - x_0|^{1-n/p} \Big(\int_G |u_x - A|^p \Big)^{1/p},$$

it being assumed that x is near enough to x_0 so that G is in R. Since x_0 is in E the integral on the right is $o(|x - x_0|^n)$, whence

$$|w(x) - w(x_0)| = o(|x - x_0|).$$

This is precisely the condition that u have a differential at x_0. Q.E.D.

Remark. Since a Lipschitz-continuous function is strongly differentiable and its derivative belongs to L_∞, we obtain as a corollary of Theorem 5 that every Lipschitz continuous function has a differential at almost all points of its domain of definition (Rademacher's theorem).

9. Two examples

Consider a set of points Q_k, $k = 1, 2, 3, \ldots$, dense on some interval I. Let a_k be a sequence of positive numbers such that $\sum a_k < \infty$, and define $f_k(x) = \log\log(1 + 1/r_k)$, where r_k denotes the distance from x to Q_k. Finally set

$$u(x) = \sum a_k f_k(x). \tag{17}$$

Since f_k is summable, it is clear from Lebesgue's theorem that the partial sums $u_m(x)$ of the series (17) converge in L_1 to $u(x)$; this shows, by the way, that u is finite almost everywhere. Moreover, since

$$\|f_k\| = \Big(\int_I |f_{kx}|^n \, dx \Big)^{1/n} < \infty,$$

we see that the sequence $\{u_{mx}\}$ is bounded, and hence weakly compact, in L_n. We let ψ denote a weak limit of this sequence. Then from

$$\int u_m \varphi_x \, dx = -\int \varphi\, u_{mx} \, dx,$$

it follows by letting $m \to \infty$ through an appropriate subsequence that

$$\int u \varphi_x \, dx = -\int \varphi \psi \, dx.$$

Thus u is strongly differentiable on I, and its derivative is in L_n. But u is infinite on a dense set, and so is everywhere discontinuous.

We shall next construct a continuous function u which has strong derivatives in L_n, but which nevertheless is nowhere differentiable. (In the paper quoted earlier, CESARI proved the existence of such a function in the case $n = 2$, but he did not give an explicit example.) The method is an elaboration and extension of one due to STEPANOFF.

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[5] See [*14*, p. 65]. The work here refers only to the one-dimensional case, but carries over to n dimensions in the obvious way.

The function to be constructed will again have the basic form $u=\sum a_k f_k$ but will be more complicated in nature. We set

$$a_k = k/2^k, \qquad N_k = 2^k, \qquad \text{and} \qquad \omega(t) = \log\frac{t+\varepsilon}{1+\varepsilon}\Big/\log\frac{\varepsilon}{1+\varepsilon},$$

where $\varepsilon=\varepsilon_k$ is fixed by the condition $\log\varepsilon = -2^k$. Also let ϱ_k be a positive number less than 2^{-2k}, to be fixed later. In order to define $f_k(x)$ we partition the interval I, which may be assumed to be a cube of unit length on each of its sides, into N_k^n subcubes each having side length 2^{-k}. Let $x_{(k)}$ be a generic symbol for the center of any one of these subcubes. We then set

$$f_k(x) = \omega\left(|x - x_{(k)}|/\varrho_k\right)$$

if $|x-x_{(k)}| \leqq \varrho_k$ for some center $x_{(k)}$, and otherwise put $f_k(x)=0$. (In geometrical language, the surface f_k consists of a family of "cones" of height one and base ϱ_k, centered at the lattice points $x_{(k)}$; in two dimensions this is an Indian mystic's bed of nails.)

Since $0 \leqq f_k \leqq 1$, it is clear that the series $\sum a_k f_k$ converges uniformly to a continuous function u. Moreover u is strongly differentiable. Indeed, the partial sums u_m satisfy

$$\|u_m\| = \left\|\sum_1^m a_k f_k\right\| \leq \sum_1^m a_k \|f_k\|$$

by Minkowski's inequality. But

$$\begin{aligned}\|f_k\| = N_k\|\omega\| &= \text{Const.}\, N_k\left(\int_0^1 |\omega'(t)|^n t^{n-1}\,dt\right)^{1/n}\\ &\leqq \text{Const.}\, N_k\,|\log\varepsilon|^{(1-n)/n},\end{aligned}$$

as one easily sees. Consequently $\|u_m\| \leqq \text{Const.}\sum k\,2^{k(1-n)/n}$, so that $\{u_{mx}\}$ forms a uniformly bounded sequence in L_n. By our earlier argument, then, u is strongly differentiable with u_x belonging to L_n.

It remains to show that u cannot have a differential at any point. Consider first a point ξ where all the functions f_k vanish. This point lies no farther than $\sqrt{n}/2^{k+1}$ from some vertex $x_{(k)}$. Hence

$$\frac{u(x_{(k)}) - u(\xi)}{|x_{(k)} - \xi|} \geqq \frac{a_k f_k(x_{(k)})}{\sqrt{n}/2^{k+1}} = \frac{2k}{\sqrt{n}}.$$

Since the right-hand side can be arbitrarily large for appropriate choice of $x_{(k)}$, it is evident that u cannot have a differential at ξ. Next, if only a finite number of functions f_k are non-zero at ξ, an easy modification of the above argument again shows that u cannot have a differential at ξ.

Finally, suppose that $f_k(\xi) \neq 0$ for an infinite number of values of k. Consider a fixed k for which $f_k(\xi) \neq 0$, and form the difference quotient with respect to the nearest of the vertices $x_{(k)}$, or else (if $|x_{(k)} - \xi| < \frac{1}{2}\varrho_k$) with respect to an appropriate point $y_{(k)}$ where $|x_{(k)} - y_{(k)}| = \varrho_k$. Then, provided the radii ϱ_k have been chosen sufficiently small, one sees that the dominant contribution to the difference quotient is due to the function $a_k f_k$. But the slope of this function becomes arbitrarily large as k tends to infinity; hence again u cannot have a differential at ξ. Thus u does not have a differential at any point in I.

Note. After this paper was completed, the author learned that C. Goffman and R. E. Hughes had independently established that a function of bounded variation in the sense of Cesari is essentially differentiable. Their proof (which will be published elsewhere) and the work of Section 3 of this paper are closely similar.

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Il Teorema del Massimo Modulo per l'Equazione dell'Elastostatica Tridimensionale

Dedicata ad ANTONIO SIGNORINI nel Suo 70° compleanno

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Sia A un campo (insieme aperto) dello spazio cartesiano delle tre variabili reali x_1, x_2, x_3; considerata in esso l'equazione vettoriale omogenea dell'equilibrio elastico

$$(1) \qquad \Delta_2 v + k \operatorname{grad} \operatorname{div} v = 0$$

$\left(\Delta_2 \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\right.$, v vettore a tre componenti reali; k costante reale maggiore di $\left.-1\right)$, ci proponiamo, sotto opportune ipotesi per la frontiera Σ di A, di dimostrare l'esistenza e pervenire al calcolo di una costante H dipendente unicamente da A e da k, tale che per ogni soluzione della (1) di classe due in A e continua in $\bar{A} = A \cup \Sigma$ riesca:

$$(2) \qquad \max_{\bar{A}} |v(x)| \leqq H \max_{\Sigma} |v(x)| .$$

Risultati come quello testè enunciato sono classicamente noti per le equazioni del secondo ordine di tipo ellittico in una funzione scalare e si conoscono gli analoghi teoremi per l'equazione parabolica del tipo del calore. Recentemente si sono ottenute delle estensioni alle più generali equazioni di tipo misto ellittico-parabolico [*2*], [*3*]. MIRANDA ha considerato alcuni anni addietro l'equazione biarmonica ed ha conseguito per essa un *teorema di massimo modulo* [*10*], e, più recentemente, ha esteso il suo risultato ad un'equazione ellittica di ordine $2m$ in due variabili [*11*]. AGMON [*1*] ha quindi considerato il caso di equazioni ellittiche in più variabili.

Occorre però dire che in queste ricerche non è stato preso in considerazione uno degli aspetti di maggior rilievo del problema, specie dal punto di vista delle applicazioni, cioè il calcolo effettivo delle costanti, analoghe alla H che interviene nella (1). Tale questione, che non si pone per le equazioni ellittiche o paraboliche del secondo ordine in una funzione scalare, riuscendo allora $H = 1$, si presenta assai complessa negli altri casi. Per quanto riguarda la (1) è stato solo osservato da G. PÓLYA [*13*] che per $k \neq 0$ non può essere $H = 1$. Inoltre è stato conseguito [*4*] il calcolo di una costante H_C tale che per ogni soluzione continua della (1) sussista la seguente maggiorazione

$$\max_{C} |v(x)| \leqq H_C \max_{\Sigma} |v(x)| ,$$

essendo C un qualsiasi compatto contenuto in A. Tuttavia la H_C ottenuta è tale che per $C \to A$ riesce $\lim H_C = +\infty$.

1. Ipotesi, premesse e tappe della dimostrazione

Supporremo che A sia un campo limitato e connesso la cui frontiera Σ sia costituita da $\nu+1$ superficie regolari chiuse: Σ_0 (contorno esterno) e $\Sigma_1, \ldots, \Sigma_\nu$ (contorni interni) a due a due disgiunte, ciascuna di classe C_h^1. Con ciò s'intende che ogni Σ_i è dotata di piano tangente in ogni punto e che esistono tre numeri positivi ϱ_A, c, h $(0<h\leqq 1)$ tali che, fissato comunque un punto x di Σ, accade che: *i*) l'intersezione $\Omega_{\varrho_A}(x)$ di Σ con la sfera di centro x e raggio ϱ_A è un insieme connesso; *ii*) assunti come asse x_3 la normale interna a Σ in x e come piano $x_1\,x_2$ il piano tangente, l'insieme $\Omega_{\varrho_A}(x)$ è una porzione di superficie di equazione $x_3=\varphi(x_1, x_2)$ con $\varphi(x_1, x_2)$ definita in un dominio T_x[1] del piano $x_1\,x_2$ (proiezione ortogonale di $\Omega_{\varrho_A}(x)$ sul piano tangente); *iii*) la funzione $\varphi(x_1, x_2)$ ha derivate prime verificanti una condizione di Hölder uniforme in T_x con coefficiente c ed esponente h.

Se B è un insieme dello spazio cartesiano a tre dimensioni, dotato di punti interni, con $C^m(B)$ indicheremo la classe delle funzioni reali di classe m in B, cioè dotate di tutte le derivate parziali fino all'ordine m, ciascuna delle quali coincide con una funzione continua in B. Dicendo che una funzione vettoriale appartiene a $C^m(B)$ intendiamo che ciascuna delle sue componenti è una funzione di $C^m(B)$.

Per ogni vettore $u(x)=\{u_1(x), u_2(x), u_3(x)\}$ della classe $C^1(\bar{A})$, detto n il versore normale interno a Σ e indicata con $\partial/\partial n$ l'operazione di derivazione secondo tale versore normale, porremo:

$$L(u)=\left[\left(k-\frac{k}{2+k}\right)\operatorname{div} u\right] n+\left(1+\frac{k}{2+k}\right)\frac{\partial u}{\partial n}+\frac{k}{2+k}\,(n\wedge\operatorname{rot} u)$$

$[(a\wedge b)$ denota il prodotto esterno dei due vettori a e $b]$; *k è una costante reale maggiore di -1.*

Porremo anche:

$$E(u)=\Delta_2 u+k \operatorname{grad}\operatorname{div} u\,.$$

Diciamo $\mathfrak{U}$ la classe dei vettori reali a tre componenti appartenenti a $C^1(\bar{A})\cap C^2(A)$, nulli su Σ e tali che $E(u)$ appartenga a $C^0(\bar{A})$. Denoteremo con $\mathscr{L}^\infty(A)$ $[\mathscr{L}^\infty(\Sigma)]$ la classe dei vettori $w(x)$ tali che $|w(x)|$ sia pseudo-limitato in A (su Σ), cioè tale che, tolti i punti di un insieme di misura nulla in A (su Σ, rispetto alla misura superficiale), la $|w(x)|$ sia limitata. Indicheremo con $\underset{A}{\mathrm{p.e.s}}\,|w(x)|$ $[\underset{\Sigma}{\mathrm{p.e.s}}\,|w(x)|]$ lo *pseudo-estremo superiore* di $w(x)$, cioè l'estremo inferiore dei numeri Q tali che $|w(x)|\leqq Q$ quasi ovunque in A (su Σ).

Consideriamo per ogni $u\in\mathfrak{U}$ l'equazione

$$\int_\Sigma \varphi\, L(u)\, d\sigma=\int_A + E(u)\, d x \tag{1.1}$$

$(d\sigma=$ misura elemento d'area su Σ, $d x=d x_1\, d x_2\, d x_3)$.

Supponiamo che φ sia assegnata in $\mathscr{L}^\infty(\Sigma)$ e si ricerca in $\mathscr{L}^\infty(A)$ una ψ incognita verificante le (1.1) per ogni $u\in\mathfrak{U}$. Tale ψ esiste comunque si assegni

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[1] Per dominio intendiamo la chiusura di un insieme aperto.

φ se e solo se esiste $H > 0$ tale che per ogni $u \in \mathfrak{U}$ riesca

$$\int_{\Sigma} |L(u)|\, d\sigma \leqq H \int_{A} |E(u)|\, dx\,.\tag{1.2}$$[2]

Se sussiste la (1.2), esiste almeno una ψ tale che

$$\operatorname*{p.e.s}_{A} |\psi| \leqq H \operatorname*{p.e.s}_{\Sigma} |\varphi|\,.\tag{1.3}$$

Se proviamo che la soluzione ψ della (1.1) è unica, assumendo quale φ la traccia su Σ di una soluzione v della (1) appartenente a $C^1(\bar{A}) \cap C^2(A)$, deve necessariamente essere (per l'identità di Green) $\psi = -v$ e quindi la (1.3) altro non è che la (2) dell'introduzione. Per provare tale diseguaglianza per le soluzioni appartenenti a $C^0(\bar{A}) \cap C^2(A)$, l'idea che si presenta spontaneamente è quella di approssimare uniformemente in $\bar{A}$ una soluzione siffatta con una successione di soluzioni appartenenti a $C^1(\bar{A}) \cap C^2(A)$. Occorrerebbe a tale proposito un teorema di densità rispetto alla convergenza uniforme delle soluzioni di classe uno in $\bar{A}$ nello spazio delle soluzioni della (1) ivi continue. Questo teorema sussiste se si riesce a provare che ogni soluzione della (1) appartenente a $C^0(\bar{A}) \cap C^2(A)$, nulla su Σ, è identicamente nulla. Tale circostanza sarà da noi dimostrata. D'altra parte, per conseguirla, si proverà che sussiste per ogni soluzione v della (1) continua in $\bar{A}$ ed ogni $u \in \mathfrak{U}$, l'identità

$$\int_{\Sigma} v\, L(u)\, d\sigma = -\int_{A} v\, E(u)\, dx\,,\tag{1.4}$$

talchè la (2) si potrà dedurre direttamente dalla (1.2) tramite la (1.4), come nel caso delle soluzioni di classe uno, senza far ricorso al teorema di densità anzidetto.

È opportuno indicare i varî punti dei quali consterà la dimostrazione della (2). Avvertiamo che con u indicheremo costantemente un elemento di $\mathfrak{U}$.

I) Esistenza e calcolo di due costanti positive H_1 e H_2 tali che:

$$\max_{\Sigma} |L(u)| \leqq H_1 \max_{\bar{A}} |E(u)| + H_2 \Big(\int_{A} |u|^2 dx \Big)^{\frac{1}{2}}.\tag{1.5}$$

II) Dimostrazione della diseguaglianza:

$$\Big(\int_{A} |u|^2 dx \Big)^{\frac{1}{2}} \leqq H_3 \Big(\int_{A} |E(u)|^2 dx \Big)^{\frac{1}{2}}\tag{1.6}$$

con calcolo esplicito per la costante positiva H_3 e, quindi, esistenza e calcolo di $H_4 > 0$ tale che

$$\max_{\Sigma} |L(u)| \leqq H_4 \max_{\bar{A}} |E(u)|\,.\tag{1.7}$$

III) Esistenza e calcolo di $H_7 > 0$ tale che:

$$\max_{\bar{A}} |u| \leqq H_7 \max_{\bar{A}} |E(u)|\,.\tag{1.8}$$

La (1.8) costituisce un teorema di massimo modulo per le soluzioni dell'equazione $E(u) = f$ non omogenea con la condizione omogenea al contorno: $u = 0$ su Σ.

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[2] *Cfr.* [2].

IV) Dimostrazione della completezza nello spazio $C^0(\bar{A})$ (munito della convergenza uniforme) del sistema di tutti i vettori $\{E(u)\}$ e conseguente deduzione (per dualità dalla (1.8)) della diseguaglianza

$$\int_A |u|\, dx \leqq H_7 \int_A |E(u)|\, dx. \tag{1.9}$$

V) Esistenza e calcolo di due costanti positive H_8 e H_9 tali che

$$\int_\Sigma |L(u)|\, d\sigma \leqq H_8 \int_A |E(u)|\, dx + H_9 \int_A |u|\, dx, \tag{1.10}$$

e conseguente deduzione della (1.2) da (1.10) e (1.9).

VI) Dimostrazione della (1.4) per ogni soluzione della (1) appartenente a $C^0(\bar{A}) \cap C^2(A)$ e conseguente deduzione della (2).

2. Dimostrazione del punto I

Poniamo $a_n = -4(2n+3)(2n+5)(2n+7)$ essendo n un intero positivo arbitrario, e

$$\psi(t) = a_n (2t-1)^{2n+1}, \qquad F(t) = \int_1^t \frac{(t-\tau)^2}{2}\, \tau (1-\tau)^3 \psi(t)\, d\tau.$$

Riesce:

$$F(0) = 0, \quad F'(0) = 1, \quad F'''(0) = 0, \quad F^{(i)}(1) = 0 \qquad (i = 0, 1, \ldots, 5),$$

$$\int_0^1 t |F'''(t)|\, dt = \frac{(2n+3)(2n+5)(2n+7)}{8(n+1)(n+2)(n+3)} \equiv b_n \tag{2.1}$$

$$\lim_{n\to\infty} b_n = 1.$$

Sia:

$$F_i = \max_{(0,1)} |F^{(i)}(t)|, \qquad \Phi_h = \max\{F_1, \ldots, F_h\}. \tag{2.2}$$

Sia ε un fissato numero positivo e, in corrispondenza ad esso, poniamo: $f(r) = \varepsilon F(r/\varepsilon)$ per $0 \leq r \leq \varepsilon$, $f(r) = 0$ per $r > \varepsilon$. Si verifica che $f(0) = 0$, $f'(0) = 1$, $f'''(0) = 0$, $f^{(i)}(\varepsilon) = 0$ $(i = 0, 1, 2, 3, 4, 5)$ e quindi $f \in C^5(0, +\infty)$. Riesce inoltre:

$$|f^{(i)}(r)| \leqq \varepsilon^{1-i} F_i, \qquad \int_0^\varepsilon |f'''(r)|\, r\, dr = b_n. \tag{2.3}$$

Siano $x \equiv (x_1, x_2, x_3)$ e $y \equiv (y_1, y_2, y_3)$ due punti dello spazio. Posto $r = |x-y|$, si ha per $r > 0$:

$$\frac{\partial^2 f(r)}{\partial x_i\, \partial x_h} = f''(r) \frac{(x_i - y_i)(x_h - y_h)}{r^2} + f'(r) \left[\frac{\delta_h^i}{r} - \frac{(x_i - y_i)(x_h - y_h)}{r^3} \right],$$

$$\Delta_2 f(r) = f''(r) + \frac{2}{r} f'(r),$$

$$\Delta_2 \Delta_2 f(r) = f^{\mathrm{IV}}(r) + \frac{4}{r} f'''(r) = f^{\mathrm{IV}}(r) + 4 \int_0^1 f^{\mathrm{IV}}(t r)\, dt = g(r).$$

Poniamo:

$$E^i_j = \delta^i_j \Delta_2 + k \frac{\partial^2}{\partial x_i\, \partial x_j}, \qquad \alpha_k = \frac{k}{2(1+k)},$$

$$\widetilde{E}^i_j = \frac{1}{2} \delta^i_j \Delta_2 - \alpha_k \frac{\partial^2}{\partial x_i\, \partial x_j}, \quad p^i_j(x, y) = \widetilde{E}^i_j[f(r)].$$

Indichiamo con $S(x, y)$ la matrice 3×3 il cui elemento generico è la funzione $s^i_j(x, y) = \widetilde{E}^i_j(r)$. Essa è una matrice *soluzione fondamentale* per l'equazione (1) ed è nota col nome di *matrice di Somigliana*[3]. Si ha $p^i_j(x, y) = s^i_j(x, y) + \tilde{s}^i_j(x, y)$ con $\tilde{s}^i_j(x, y) = O(1)$.

Sia $p^i(x, y)$ il vettore di componenti $p^i_1(x, y)$, $p^i_2(x, y)$, $p^i_3(x, y)$ e sia L_h $(h = 1, 2, 3)$ la h-esima componente dell'operatore differenziale al contorno L sopra introdotto. Si ha:

$$\begin{aligned} L^x_h(p^i) &= \frac{\delta^i_h}{2} \frac{\partial}{\partial n^x} \Delta_2 f - \frac{k}{k+2} \frac{\partial}{\partial n^x} \frac{\partial^2 f}{\partial x_i\, \partial x_h} + \frac{1}{2} \frac{k}{k+2} \left(\frac{\partial \Delta_2 f}{\partial x_i} n_h + \frac{\partial \Delta_2 f}{\partial x_h} n_i \right) \\ &= \delta^i_h \left[\frac{1}{2} \frac{\partial}{\partial n^x} \Delta_2 f - \frac{k}{k+2} \frac{\partial}{\partial n^x} \frac{f'}{r} \right] + \\ &\quad + \frac{k}{k+2} \left[- f''' \frac{\partial r}{\partial n^x} \frac{(x_i - y_i)(x_h - y_h)}{r^2} - f''(x_i - y_i)(x_h - y_h) \frac{\partial}{\partial n^x} \frac{1}{r^2} + \right. \\ &\quad + f'' \frac{\partial r}{\partial n^x} \frac{(x_i - y_i)(x_h - y_h)}{r^3} + f'(x_i - y_i)(x_h - y_h) \frac{\partial}{\partial n^x} \frac{1}{r^3} + \\ &\quad \left. + \frac{1}{2} f''' \frac{(x_i - y_i) n_h + (x_h - y_h) n_i}{r} \right].^{4} \end{aligned} \tag{2.4}$$

Si ha d'altra parte per ogni punto ξ di A:

$$- 4\pi u_i(\xi) = \int_\Sigma L(u)\, p^i(\xi, y)\, d\sigma_y + \int_A E(u)\, p^i(\xi, y)\, dy - \int_A u\, E[p^i(\xi, y)]\, dy.$$

Posto $E^j[w] = E^j_s[w_s]$ $[w \equiv (w_1, w_2, w_3)]$, si ha $E^j[p^i(x, y)] = \frac{1}{2} \delta^i_j \Delta^2_2 f(r)$, sicchè, posto $g^i \equiv (\delta^i_1 \Delta^2_2 f, \delta^i_2 \Delta^2_2 f, \delta^i_3 \Delta^2_2 f)$, si ha:

$$- 4\pi u_i(\xi) = \int_\Sigma L(u)\, p^i(\xi, y)\, d\sigma_y + \int_A E(u)\, p^i(\xi, y)\, dy - \tfrac{1}{2} \int_A u\, g^i(\xi, y)\, dy.$$

Sia x un punto di Σ, n^x la normale (interna) a Σ in x, ξ un punto di A su n^x. Applicando ad ambo i membri della precedente relazione l'operatore L relativo alla normale n^x e facendo tendere ξ a x su n^x si trova (*cfr.* [5] teor. XIII) per ogni $x \in \Sigma$:

$$- 2\pi L(u) = \int_\Sigma L(u) * \{L^x_h(p^i)\}\, d\sigma_y + \int_A E(u) * \{L^x_h(p^i)\}\, dy - \tfrac{1}{2} \int_A u * \{L^x_h(g^i)\}\, dy.^{5} \tag{2.5}$$

Ne viene:

$$\begin{aligned} 2\pi |L(u)| \leqq \max_\Sigma |L(u)| \int_\Sigma \| L^x_h(p^i) \|\, d\sigma_y &+ \\ + \max_A |E(u)| \int_A \| L^x_h(p^i) \|\, dy + \tfrac{1}{2} \int_A |u|\, \| L^x_h(g^i) \|\, dy&. \end{aligned} \tag{2.6}$$

[3] *Cfr.* [5].

[4] Scrivendo L^x si intende che l'operatore L opera su $p^i(x, y)$ come funzione di x; con n^x si intende la normale interna a Σ in x; n_1, n_2, n_3 sono le componenti del versore normale interno a Σ.

[5] Con il simbolo $w * \{a^i_j\}$ indichiamo il vettore di componenti $a^1_j u_j$, $a^2_j u_j$, $a^3_j u_j$. Porremo inoltre $\| a^i_j \| = \left[\sum_{i,j} (a^i_j)^2 \right]^{\frac{1}{2}}$.

Supponiamo, cosa ben lecita, che il numero ϱ_A, in precedenza introdotto quale costante geometrica legata ad A, sia minore di 1. Assumiamo $\varepsilon < \varrho_A$. Sia $\Omega_\varepsilon(x)$ l'intersezione di Σ con la sfera chiusa $I_\varepsilon(x)$ di centro x e raggio ε. Poichè i vettori p^i hanno, come funzioni di y, supporto contenuto in $I_\varepsilon(x)$, si ha:

$$\int_\Sigma \| L_h^x(p^i) \| \, d\sigma_y = \int_{\Omega_\varepsilon(x)} \| L_h^x(p^i) \| \, d\sigma_y, \qquad \int_A \| L_h^x(p^i) \| \, dy = \int_{I_\varepsilon(x) \cap A} \| L_h^x(p^i) \| \, dy .$$

Per la (2.4) si ottiene:

$$\begin{aligned} \int_\Sigma \| L_h^x(p^i) \| \, d\sigma_y &\leqq \sqrt{3} \int_{\Omega_\varepsilon(x)} \left| \frac{1}{2} \frac{\partial}{\partial n^x} \Delta_2 f - \frac{k}{k+2} \frac{\partial}{\partial n^x} \frac{f'}{r} \right| d\sigma_y + \\ &+ \frac{|k|}{k+2} \int_{\Omega_\varepsilon(x)} \left[|f'''| \left| \frac{\partial r}{\partial n^x} \right| + |f''| \, r^2 \left| \frac{\partial}{\partial n^x} \frac{1}{r^2} \right| + |f''| \left| \frac{\partial r}{\partial n^x} \right| \frac{1}{r} + |f'| \, r^2 \left| \frac{\partial}{\partial n^x} \frac{1}{r^3} \right| \right] d\sigma_y + \\ &+ \frac{|k|}{k+2} \int_{\Omega_\varepsilon(x)} |f'''| \, d\sigma_y \leqq \left(\frac{\sqrt{3}}{2} + \frac{|k|}{k+2} \right) \int_{\Omega_\varepsilon(x)} |f'''| \left| \frac{\partial r}{\partial n^x} \right| d\sigma_y + \\ &+ \frac{2\sqrt{3} + 3|k|}{k+2} \left(\int_{\Omega_\varepsilon(x)} |f''| \frac{1}{r} \left| \frac{\partial r}{\partial n^x} \right| d\sigma_y + \int_{\Omega_\varepsilon(x)} |f'| \frac{1}{r^2} \left| \frac{\partial r}{\partial n^x} \right| d\sigma_y \right) + \frac{|k|}{k+2} \int_{\Omega_\varepsilon(x)} |f'''| \, d\sigma_y . \end{aligned}$$

Si ha:

$$\begin{aligned} \left| \frac{\partial r}{\partial n^x} \right| &= \left| \frac{(x_1 - y_1)\, \varphi_{x_1} + (x_2 - y_2)\, \varphi_{x_2} - (x_3 - y_3)}{r} \right| (1 + \varphi_{x_1} + \varphi_{x_2})^{-\frac{1}{2}} \\ &\leqq \frac{1}{r} \left| (x_1 - y_1)\, \varphi_{x_1} + (x_2 - y_2)\, \varphi_{x_2} - \varphi(x_1, x_2) + \varphi(y_1, y_2) \right| \\ &= r^{-1} \left| (x_1 - y_1)\, \varphi_{x_1} + (x_2 - y_2)\, \varphi_{x_2} - \varphi_{x_2}[x_1, x_2 + \vartheta(y_2 - x_2)]\, (x_2 - y_2) - \right. \\ &\qquad \left. - \varphi_{x_1}[x_1 + \vartheta'(y_1 - x_1), y_2]\, (x_1 - y_1) \right| \\ &\leqq r^{-1} |x_1 - y_1| \, c\, r^h + r^{-1} |x_2 - y_2| \, c\, r^h \leqq 2 c\, r^h . \end{aligned}$$

Sia ϱ, ϑ un sistema di coordinate polari nel piano tangente a Σ in x, avente polo in x. Si ha:

$$d\sigma_y = (1 + \varphi_{y_1}^2 + \varphi_{y_2}^2)^{\frac{1}{2}} \, dy_1 \, dy_2 \leqq \sqrt{1 + 2c^2 \varrho^{2h}} \, \varrho \, d\varrho \, d\vartheta \leqq (1 + \sqrt{2}\, c\, \varrho^h)\, \varrho \, d\varrho \, d\vartheta$$

e pertanto, tenendo presente la prima delle (2.3):

$$\begin{aligned} \int_\Sigma \| L_h^x(p^i) \| \, d\sigma_y &\leqq 4\pi c (1 + \sqrt{2} c) \left(\frac{\sqrt{3}}{2} + \frac{|k|}{k+2} \right) \frac{F_3}{\varepsilon^2} \int_0^\varepsilon \varrho^{h+1} \, d\varrho + \\ &+ 4\pi c (1 + \sqrt{2} c) \frac{2\sqrt{3} + 3|k|}{k+2} \left(\frac{F_2}{\varepsilon} \int_0^\varepsilon \varrho^h \, d\varrho + F_1 \int_0^\varepsilon \varrho^{h-1} \, d\varrho \right) + \\ &+ 2\sqrt{2}\, \pi c \frac{|k|}{k+2} \frac{F_3}{\varepsilon^2} \int_0^\varepsilon \varrho^{h+1} \, d\varrho + 2\pi \frac{|k|}{k+2} \int_0^\varepsilon |f'''(\varrho)| \, \varrho \, d\varrho \\ &= \left[4\pi c (1 + \sqrt{2} c) \left(\frac{\sqrt{3}}{2} + \frac{|k|}{k+2} \right) + 2\sqrt{2}\, \pi c \frac{|k|}{k+2} \right] \frac{\varepsilon^h}{2+h} F_3 + \\ &+ 4\pi c (1 + \sqrt{2} c) \frac{2\sqrt{3} + 3|k|}{k+2} \left(\frac{\varepsilon^h}{1+h} F_2 + \frac{\varepsilon^h}{h} F_1 \right) + 2\pi \frac{|k|}{k+2} \int_0^\varepsilon |f'''(\varrho)| \, \varrho \, d\varrho . \end{aligned}$$

Ne segue, ricordando la seconda delle (2.2)

$$\int_{\Sigma} \| L_h^x(p^i) \| \, d\sigma_y \leqq M_1 \, \Phi_3 \, \varepsilon^h + \frac{2\pi|k|}{k+2} \int_0^{\varepsilon} |f'''(\varrho)| \, \varrho \, d\varrho \tag{2.7}$$

con

$$\begin{aligned} M_1 = 4\pi c \Big\{ \Big[(1+\sqrt{2}c) \Big(\frac{\sqrt{3}}{2} + \frac{|k|}{k+2} \Big) + \frac{1}{\sqrt{2}} \frac{|k|}{k+2} \Big] \frac{1}{2+h} + \\ + (1+\sqrt{2}c) \frac{2\sqrt{3}+3|k|}{k+2} \Big(\frac{1}{h} + \frac{1}{h+1} \Big) \Big\}. \end{aligned} \tag{2.8}$$

Posto $A_\varepsilon(x) = I_\varepsilon(x) \cap A$ per la (2.4) riesce:

$$\begin{aligned} \int_A \| L_h^x(p^i) \| \, dy \leqq \Big(\frac{\sqrt{3}}{2} + \frac{|k|}{k+2} \Big) \int_{A_\varepsilon(x)} |f'''| \, \Big| \frac{\partial r}{\partial n^x} \Big| \, dy + \\ + \frac{2\sqrt{3}+3|k|}{k+2} \cdot \Big(\int_{A_\varepsilon(x)} |f''| \frac{1}{r} \Big| \frac{\partial r}{\partial n^x} \Big| \, dy + \int_{A_\varepsilon(x)} |f'| \frac{1}{r^2} \Big| \frac{\partial r}{\partial n^x} \Big| \, dy \Big) + \frac{|k|}{k+2} \int_{A_\varepsilon(x)} |f'''| \, dy, \end{aligned}$$

da questa, tenendo presente che

$$\Big| \frac{\partial r}{\partial n^x} \Big| \leqq 1, \qquad \int_{A_\varepsilon(x)} \frac{1}{r} \, dy \leqq 2\pi \varepsilon^2, \qquad \int_{A_\varepsilon(x)} \frac{1}{r^2} \, dy \leqq 4\pi \varepsilon,$$

si trae:

$$\begin{aligned} \int_A \| L_h^x(p^i) \| \, dy \leqq \frac{4}{3} \pi \Big(\frac{\sqrt{3}}{2} + \frac{2|k|}{k+2} \Big) F_3 \, \varepsilon + \\ + 2\pi \frac{2\sqrt{3}+3|k|}{k+2} (F_2 + 2F_1) \, \varepsilon \leqq \frac{2\pi}{3} \Phi_3 \, \varepsilon \, \frac{\sqrt{3}(k+2) + 31|k| + 18\sqrt{3}}{k+2}. \end{aligned}$$

Quindi:

$$\int_A \| L_h^x(p^i) \| \, dy \leqq M_2, \tag{2.9}$$

con

$$M_2 = \frac{2\pi}{3} \frac{\sqrt{3}(k+2) + 31|k| + 18\sqrt{3}}{k+2} \Phi_3 \, \varepsilon. \tag{2.10}$$

Si ha:

$$\begin{aligned} \| L_h^x(g^i) \| \leqq \Big(\sqrt{3} + |k| + \frac{\sqrt{2}|k|}{k+2} \Big) |g'(r)| \\ = \Big(\sqrt{3} + |k| + \frac{\sqrt{2}|k|}{k+2} \Big) \Big| f^{\mathrm{V}}(r) + 4 \int_0^1 t \, f^{\mathrm{V}}(rt) \, dt \Big| \leqq 3 \Big(\sqrt{3} + |k| + \frac{\sqrt{2}|k|}{k+2} \Big) \frac{F_5}{\varepsilon^4}, \end{aligned}$$

donde

$$\begin{aligned} \frac{1}{2} \int_A |u| \, \| L_h^x(g^i) \| \, dy \leqq \frac{1}{2} \Big(\int_{A_\varepsilon(x)} \| L_h^x(g^i) \|^2 \, dy \Big)^{\frac{1}{2}} \Big(\int_A |u|^2 \, dy \Big)^{\frac{1}{2}} \\ \leqq \sqrt{3\pi} \Big(\sqrt{3} + |k| + \frac{\sqrt{2}|k|}{k+2} \Big) F_5 \, \varepsilon^{-\frac{5}{2}} \Big(\int_A |u|^2 \, dy \Big)^{\frac{1}{2}}. \end{aligned}$$

Ne viene

$$M_3 \Big(\int_A |u|^2 \, dy \Big)^{\frac{1}{2}} \geqq \tfrac{1}{2} \int_A |u| \, \| L_h^x(g^i) \| \, dy \tag{2.11}$$

con

$$M_3 = \sqrt{3\pi} \Big(\sqrt{3} + |k| + \frac{\sqrt{2}|k|}{k+2} \Big) F_5 \, \varepsilon^{-\frac{5}{2}}. \tag{2.12}$$

Da (2.6), (2.7), (2.9), (2.11) si trae

$$
\begin{aligned}
2\pi \max_{\Sigma} |L(u)| &\leqq M_1 \Phi_3 \varepsilon^h \max_{\Sigma} |L(u)| + \\
&+ \frac{2\pi|k|}{k+2}\left(\int_0^{\varepsilon} |f'''|\, \varrho\, d\varrho\right) \max_{\Sigma} |L(u)| + M_2 \max_{\bar{A}} |E(u)| + M_3\left(\int_A |u|^2 dy\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.13}
$$

Tenendo presente la seconda delle (2.3) e la (2.1), si può assumere n tale che

$$
1 - \frac{|k|}{2+k} b_n > \frac{1}{2}\left(1 - \frac{|k|}{2+k}\right), \qquad M_1 |a_n| > 2\pi\left(1 - \frac{|k|}{2+k} b_n\right). \tag{2.14}
$$

Una volta fissato n, nel modo indicato, si scelga ε tale che

$$
\varepsilon \leqq \left[\frac{2\pi\left(1 - \frac{|k|}{2+k} b_n\right)}{M_1 |a_n|}\right]^{1/h} \varrho_A, \tag{2.15}
$$

ciò che implica $\varepsilon < \varrho_A$. Si ha allora:

$$
2\pi\left(1 - \frac{|k|}{2+k} b_n\right) - M_1 |a_n|\, \varepsilon^h > 0.
$$

Poichè può assumersi $\Phi_3 = |a_n|$, dalla (2.13) si trae:

$$
\begin{aligned}
\left[2\pi\left(1 - \frac{|k|}{2+k} b_n\right) - M_1 |a_n|\, \varepsilon^h\right] \max_{\Sigma} |L(u)| \\
\leq M_2 \max_{\bar{A}} |E(u)| + M_3\left(\int_A |u|^2 dy\right)^{\frac{1}{2}}.
\end{aligned}
$$

Si deduce allora la (1.5) con

$$
H_1 = \frac{M_2}{2\pi\left(1 - \frac{|k|}{2+k} b_n\right) - M_1 |a_n|\, \varepsilon^h}; \qquad H_2 = \frac{M_3}{2\pi\left(1 - \frac{|k|}{2+k} b_n\right) - M_1 |a_n|\, \varepsilon^h} \tag{2.16}
$$

dove M_1, M_2, M_3 sono date da (2.8), (2.10), (2.12).

3. Dimostrazione del punto II

Proveremo la (1.6) come caso particolare di una formola di maggiorazione relativa a sistemi di equazioni più generali di quello dell'elastostatica[6].

In questo paragrafo A denoterà un *campo regolare* dello spazio cartesiano S_q ($q \geqq 1$), cioè tale che $\partial A = \partial \bar{A}$, essendo la frontiera ∂A costituita da porzioni di ipersuperficie regolari in guisa tale che sussistano le formole di Gauss-Green che mutano integrali estesi ad A in integrali ipersuperficiali estesi a ∂A. Sia m un intero positivo e supponiamo che le funzioni reali $a_{ij}^{hk}(x)$ ($h, k = 1, \ldots, q$; $i, j = 1, \ldots, m$) del punto $x \equiv (x_1, \ldots, x_q)$ di $\bar{A}$ appartengano a $C^2(\bar{A})$, le $b_{ij}^h(x)$ a $C^1(\bar{A})$ e le $c_{ij}(x)$ a $C^0(\bar{A})$. Poniamo:

$$
D_h = \frac{\partial}{\partial x_h}, \qquad D_{hk} = \frac{\partial^2}{\partial x_h\, \partial x_k}, \qquad E_{ij} = a_{ij}^{hk} D_{hk} + b_{ij}^h D_h + c_{ij},
$$

$$
E_{ij}^* = D_h\, a_{ij}^{hk} D_k - D_h\, b_{ij}^h + c_{ij}.
$$

---

[6] *Cfr.* [*4*], dove la maggiorazione in questione trovasi data per la prima volta. *Cfr.* anche [*12*].

Facciamo le seguenti ipotesi:

1) $a^{hk}_{ij} \equiv a^{kh}_{ij}$, $a^{hk}_{ij} \equiv a^{hk}_{ji}$, $b^h_{ij} \equiv b^h_{ji}$;

2) la forma quadratica $a^{hk}_{ij}(x)\,\lambda^h_i\,\lambda^k_j$ è semidefinita positiva per ogni $x \in \bar{A}$;

3) la forma quadratica $c_{ij}(x)\,\lambda_i\,\lambda_j$ è semidefinita negativa per ogni $x \in \bar{A}$;

4) esiste una funzione $v \in C^2(\bar{A})$ non positiva in $\bar{A}$ e tale che la forma quadratica $E^*_{ij}(v)\,\lambda_i\,\lambda_j$ è definita positiva per ogni $x \in \bar{A}$.

Per ogni coppia di funzioni v e w di $C^2(\bar{A})$ si ha:

$$\int_A [v\,E_{ij}(w) - w\,E^*_{ij}(v)]\,dx = -\int_{\partial A} [v\,(a^{hk}_{ij}\,D_h w)\,n_k - w\,(a^{hk}_{ij}\,D_h v \cdot n_k - b_{ij}\,v)]\,d\sigma, \tag{3.1}$$

essendo n_k la k-esima componente del versore normale interno a ∂A e $b_{ij} = (b^h_{ij}\quad D_k\,a^{hk}_{ij})\,n_h$. Si assuma nella (3.1) come v la funzione indicata nella ipotesi 4) e $w = u_i\,u_j$, essendo $u \equiv (u_1, \ldots, u_m)$ un vettore appartenente a $C^1(\bar{A}) \cap C^2(A)$ nullo su ∂A e tale che $E_{ij}(u_j)$ appartenga a $C^0(\bar{A})$. Poichè riesce:

$$E_{ij}(u_i\,u_j) = u_j\,E_{ij}(u_i) + u_i\,E_{ij}(u_j) + 2a^{hk}_{ij}\,D_h u_i\,D_k u_j - c_{ij}\,u_i\,u_j,$$

si ha:

$$\begin{aligned}\int_A v\,[u_j\,E_{ij}(u_i) + u_i\,E_{ij}(u_j)]\,dx + 2\int_A v\,a^{hk}_{ij}\,D_h u_i \cdot D_k u_j\,dx\,- \\ - \int_A v\,c_{ij}\,u_i\,u_j\,dx - \int_A u_i\,u_j\,E^*_{ij}(v)\,dx = 0.\end{aligned} \tag{3.2}$$

Per l'ipotesi 1) la (3.2) può scriversi:

$$2\int_A v\,u_i\,E_{ij}(u_j)\,dx + 2\int_A v\,a^{hk}_{ij}\,D_h u_i \cdot D_k u_j\,dx - \int_A v\,c_{ij}\,u_i\,u_j\,dx = \int_A u_i\,u_j\,E^*_{ij}(v)\,dx \tag{3.3}$$

e per le 1), 2), 3), 4) riesce:

$$\int_A v\,a^{hk}_{ij}\,D_h u_i \cdot D_k u_j\,dx - \int_A v\,c_{ij}\,u_i\,u_j\,dx \leqq 0.$$

Dalla (3.3) si trae allora:

$$\int_A u_i\,u_j\,E^*_{ij}(v)\,dx \leqq 2\int_A v\,u_i\,E_{ij}(u_j)\,dx.$$

Indicato con Ω il determinante della matrice $\{E^*_{ij}(v)\}$ e con Ω_{ii} il complemento algebrico dell'elemento $E^*_{ii}(v)$ di tale matrice, riesce: $E^*_{ij}(v)\,\lambda_i\,\lambda_j \geqq \Omega\,m^{-1} \sum_{i=1}^{m} \Omega^{-1}_{ii}\,\lambda^2_i$. Posto $\Omega^* = \max(\Omega_{11}, \ldots, \Omega_{mm})$, segue:

$$\begin{aligned} m^{-1} \min_{\bar{A}} \frac{\Omega}{\Omega^*} \int_A |u|^2\,dx &\leqq m^{-1} \int_A \Omega \sum_{i=1}^{m} \Omega^{-1}_{ii}\,u^2_i\,dx \\ &\leqq 2\int_A v\,u_i\,E_{ij}(u_j)\,dx \leqq 2 \max_{\bar{A}} |v| \int_A |u|\,|E(u)|\,dx,\end{aligned}$$

avendo indicato con u il vettore $(u_1, \ldots, u_m)$ e con $E(u)$ il vettore $(E_{1j}(u_j), \ldots, E_{mj}(u_j))$. Abbiamo quindi:

$$\left(\int_A |u|^2\,dx\right)^{\frac{1}{2}} \leqq 2m \max_{\bar{A}} (\Omega^*\,\Omega^{-1}) \max_{\bar{A}} |v| \left(\int_A |E(u)|^2\,dx\right)^{\frac{1}{2}}. \tag{3.4}$$

Sia ora $q=m$ e $E(u)\equiv \varDelta_2 u+k\,\mathrm{grad}\,\mathrm{div}\,u$, $(k>-1)$. Detto d il raggio di una sfera che contiene $\bar{A}$ nel suo interno e di centro x^0, si assuma $v=|x-x_0|^2-d^2$. È facile verificare che sono soddisfatte le 1), 2), 3), 4). Poichè riesce: $E_{ij}(v)=E^*_{ij}(v)=2\delta^i_j(m+k)$, $\Omega^*\,\Omega^{-1}=[2(m+k)]^{-1}$, dalla (3.4) si trae:

$$\left(\int_A |u|^2\,dx\right)^{\frac{1}{2}}\leqq \frac{m\,d^2}{m+k}\left(\int_A |E(u)|^2\,dx\right)^{\frac{1}{2}}.$$

È quindi provata la (1.6) con

$$H_3=\frac{3\,d^2}{3+k}, \tag{3.5}$$

essendo d il raggio di una qualsiasi sfera che contiene il corpo elastico.

Dalla (1.5) e dalla (1.6) si deduce la (1.7) con

$$H_4=H_1+H_2\,H_3(\mathrm{vol}\,A)^{\frac{1}{2}}. \tag{3.6}$$

4. Dimostrazione del punto III

Per ogni $x\in A$ si ha

$$-4\pi\,u(x)=\int_\Sigma L[u(y)]*\{s^i_j(x,y)\}\,d\sigma_y+\int_A E[u(y)]*\{s^i_j(x,y)\}\,dy.$$

Si trae:

$$4\pi\,|u(x)|\leqq \max_\Sigma |L(u)|\int_\Sigma \|s^i_j(x,y)\|\,d\sigma_y+\max_{\bar{A}} |E(u)|\int_A \|s^i_j(x,y)\|\,dy.$$

Le funzioni di x:

$$\frac{1}{4\pi}\int_\Sigma \|s^i_j(x,y)\|\,d\sigma_y, \qquad \frac{1}{4\pi}\int_A \|s^i_j(x,y)\|\,dy$$

sono continue in S_3; dettine H_5 ed H_6 i massimi in $\bar{A}$, sussiste la diseguaglianza:

$$\max_{\bar{A}} |u(x)|\leqq H_5\max_\Sigma |L(u)|+H_6\max_{\bar{A}} |E(u)|.$$

Da essa, tramite la (1.7), si deduce la (1.8) con

$$H_7=H_4\,H_5+H_6. \tag{4.1}$$

Per maggiorare H_5 e H_6 si osservi che

$$\begin{aligned}\|s^i_j(x,y)\|&=\left\|\delta^i_j(1-\alpha_k)\,\frac{1}{r}-\frac{(x_i-y_i)\,(x_j-y_j)}{r^3}\right\|\\ &\leqq \sqrt{3}\,|1-\alpha_k|\,\frac{1}{r}+\frac{\alpha_k}{r^3}\left[\sum_{ij}^{1,3}(x_i-y_i)^2\,(x_j-y_j)^2\right]^{\frac{1}{2}}=\frac{\sqrt{3}\,(k+2)+|k|}{2k+2}\,\frac{1}{r}.\end{aligned}$$

Detto x_0 il punto di massimo su Σ della funzione $\int_\Sigma \frac{d\sigma_y}{|x-y|}$, si ha:

$$\int_\Sigma \frac{d\sigma_y}{|x-y|}\leqq \int_{\Omega_{\varrho_A}(x)} \frac{d\sigma_y}{|x_0-y|}+\int_{\Sigma-\Omega_{\varrho_A}(x)} \frac{d\sigma_y}{|x_0-y|}\leqq 2\pi\,\varrho_A\sqrt{1+2c^2}+\frac{\text{area}\,\Sigma}{\varrho_A}.$$

Riesce d'altra parte $\int_A \frac{dy}{|x-y|} \leqq 2\pi\delta^2$, essendo δ il diametro di A, per modo che possiamo maggiorare H_5 ed H_6 al modo seguente:

$$H_5 \leqq \frac{\sqrt{3}\,(k+2)+|k|}{2k+2}\left(\frac{1}{2}\,\varrho_A\sqrt{1+2c^2}+\frac{1}{4\pi}\,\frac{\text{area }\Sigma}{\varrho_A}\right), \tag{4.2}$$

$$H_6 \leqq \frac{\sqrt{3}\,(k+2)+|k|}{2k+2}\,\frac{\delta^2}{2}\,. \tag{4.3}$$

5. Dimostrazione del punto IV

Sia $w(x) \equiv \big(w_1(x), w_2(x), w_3(x)\big)$ un vettore appartenente a $\mathfrak{H}^0(\bar{A})$[7]. In particolare, potremmo assumere come $w(x)$ un vettore le cui componenti sono polinomi. Consideriamo nella classe dei vettori continui su Σ l'equazione integrale nell'incognita φ:

$$2\pi\,\varphi(x) + \int_\Sigma \varphi(y) * \{L_j^x(s^i(x,y))\}\,d\sigma_y + \int_A w(y) * \{L_j^x[s^i(x,y)]\}\,dy = 0. \tag{5.1}$$

Poichè riesce $L_j\big(s^i(x,y)\big) = O\big(|x-y|^{h-2}\big)$ per $x, y \in \Sigma$, la (5.1) è una equazione di Fredholm (di II specie). Se la soluzione di essa esiste, appartiene a $\mathfrak{H}^0(\Sigma)$[8]. L'equazione omogenea trasposta di (5.1) è:

$$2\pi\,\tau(x) + \int_\Sigma \tau(y) * \{L_j^y(s^i(x,y))\}\,d\sigma_y = 0. \tag{5.2}$$

Si consideri, in corrispondenza della soluzione τ di (5.2), la funzione:

$$z(x) = \int_\Sigma \tau(y) * \{L_j^y(s^i(x,y))\}\,d\sigma_y;$$

essa appartiene a $\mathfrak{H}^1(\bar{A})$[9]. Poichè z è in A soluzione della equazione dell'elastostatica e dato che si annulla su Σ, essa è identicamente nulla in $\bar{A}$[10].

Poichè riesce:

$$\int_\Sigma \tau(x)\,d\sigma_x \int_A w(y) * \{L_j^x(s^i(x,y))\}\,dy = \int_A w(x)\,z(x)\,dx = 0,$$

resta provato che la (5.1) ammette soluzione.

Se φ è una soluzione di (5.1), poniamo:

$$u(x) = -\frac{1}{4\pi}\int_\Sigma \varphi(y) * \{s_j^i(x,y)\}\,d\sigma_y - \frac{1}{4\pi}\int_A w(y) * \{s_j^i(x,y)\}\,dy. \tag{5.3}$$

La funzione vettoriale $u(x)$ gode delle seguenti proprietà: 1) è continua in tutto lo spazio; 2) detto CA il complementare di A, u appartiene a $\mathfrak{H}^1(\bar{A})$ ed a $\mathfrak{H}^1(CA)$; 3) $E(u) = w$ in A; 4) $E(u) = 0$ in $C\bar{A}$ e, come funzione di $\mathfrak{H}^1(CA)$, u verifica la condizione al contorno $L(u) = 0$ su Σ; 5) $u = 0$ su Σ; 6) u ha le derivate prime con modulo di quadrato sommabile in CA; 7) è infinitesima all'infinito.

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[7] Dicendo che $w(x)$ appartiene a $\mathfrak{H}^m(B)$ $(m \geqq 0)$, intendiamo che esso è un vettore di $C^m(B)$ e le sue derivate m-esime verificano una condizione di Hölder uniforme in B.

[8] *Cfr.* [7], p. 170. La dimostrazione ivi contenuta è relativa al caso $k = 0$, tuttavia essa sussiste praticamente inalterata per ogni $k > -1$.

[9] *Cfr.* [7], p. 212. Sussiste anche in questo caso l'osservazione della nota[8].

[10] *Cfr.* [5], teor. II.

Da 5), 6) e 7) segue che u è nulla all'esterno di Σ_0[11] e, se $\nu>0$, è costante all'interno di ogni Σ_i $(i>0)$. Se è $\nu=0$, resta provato che $u\in\mathfrak{U}$. Se è $\nu>0$, esistono 3ν autosoluzioni linearmente indipendenti della (5.2); esse sono costituite dai vettori $\tau^{mj}(x)$ $(m=1,\ldots,\nu;\ j=1,2,3)$ aventi la i-esima componente τ_i^{mj} così definita:

$$\tau_i^{mj}(x)\begin{cases}=\delta_i^j & x\in\Sigma_m\\ =0 & x\in\Sigma-\Sigma_m.\end{cases}$$

Siano $\varphi^1(x),\ldots,\varphi^{3\nu}(x)$ autosoluzioni linearmente indipendenti dell'equazione omogenea associata a (5.1) e sia $\varphi^0(x)$ una ben fissata soluzione di questa equazione. Poniamo $\varphi(x)=\varphi^0(x)+\sum\limits_{m=1}^{3\nu}c_m\varphi^m(x)$, essendo le c_m costanti. Sia $u(x)$ definita da (5.3) e $u^0(x)$ dall'analoga di (5.3) ottenuta sostituendo φ con φ^0. La funzione $\tilde{u}(x)=u(x)-u^0(x)$ è nulla all'esterno di Σ_0 e costante all'interno di ogni Σ_i $(i>0)$. Non possono scegliersi le costanti c_m in guisa tale che $\tilde{u}(x)$ sia nulla all'interno di tutte le Σ_i. Infatti, nel caso contrario, sarebbe

$$\tilde{u}(x)=\int\limits_{\Sigma}\Big(\sum_{m=1}^{3\nu}c_m\varphi^m(y)\Big)*\{s_j^i(x,y)\}\,d\sigma_y=0$$

e quindi $\sum\limits_{m=1}^{3\nu}c_m\varphi^m(y)\equiv 0$ su Σ. Ne viene, scelto x^n interno a Σ_n, che la matrice:

$$\Big\{\int\limits_{\Sigma}\varphi^m(y)\,\{s_j^i(x^n,y)\}\,d\sigma_y\Big\}$$

ha determinante non nullo. Pertanto esiste una (sola) 3ν-pla di costanti c_m tali che:

$$\begin{aligned}&\sum_{m=1}^{3\nu}c_m\int\limits_{\Sigma}\varphi^m(y)*\{s_j^i(x^n,y)\}\,d\sigma_y\\&=-\int\limits_{A}w(y)*\{s_j^i(x^n,y)\}\,dy-\int\limits_{\Sigma}\varphi^0(y)*\{s_j^i(x^n,y)\}\,d\sigma_y\qquad(n=1,\ldots,\nu).\end{aligned}$$

In corrispondenza a tale scelta delle costanti è $u(x^n)=0$ $(n=1,\ldots,\nu)$ e quindi $u(x)\equiv 0$ in $C(\bar{A})$. Pertanto è $u\in\mathfrak{U}$ anche per $\nu>0$. Dalla proprietà 3) di u e attesa l'arbitrarietà di w segue la proprietà di completezza asserita nel punto IV.

È opportuno notare che se Σ è di classe C_h^2, essendo allora la $u(x)$ testè costruita appartenente alla classe $C^2(\bar{A})$, la proprietà di completezza in discorso è verificata dai vettori nulli su Σ ed appartenenti a $C^2(\bar{A})$. Tale classe, che ovviamente è contenuta in $\mathfrak{U}$, sarà da noi in seguito indicata con $\mathfrak{U}_0$.

Dopo quanto si è provato, può affermarsi il sussistere della (1.9), ottenuta per dualità dalla (1.8)[12].

6. Dimostrazione del punto V

Dalla (2.5) si trae:

$$\begin{aligned}2\pi\int\limits_{\Sigma}|L(u)|\,d\sigma\leq&\int\limits_{\Sigma}|L(u(y))|\,d\sigma_y\int\limits_{\Sigma}\|L_h^x(p^i(x,y))\|\,d\sigma_x+\\&+\int\limits_{A}|E(u(y))|\,dy\int\limits_{\Sigma}\|L_h^x(p^i(x,y))\|\,d\sigma_x+\tfrac{1}{2}\int\limits_{A}|u(y)|\,dy\int\limits_{\Sigma}\|L_h^x(g^i(x,y))\|\,d\sigma_x.\end{aligned}\tag{6.1}$$

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[11] *Cfr.* [5], teor. III.

[12] *Cfr.* [6].

Con procedimento del tutto analogo a quello seguito nel §2, da (6.1) si trae:

$$(6.2)\qquad \begin{aligned} 2\pi \int_{\Sigma} |L(u)|\, d\sigma \leqq \left[N_1\, |a_n|\, \varepsilon^h + \frac{2\pi |k|}{2+k}\, b_n\right] \int_{\Sigma} |L(u)|\, d\sigma \, + \\ + N_2 \int_{A} |E(u)|\, dy + N_3 \int_{A} |u(y)|\, dy, \end{aligned}$$

ove:

$$(6.3)\qquad N_1 = 4\pi c \left\{\frac{1}{2}\, \frac{1}{2+h}\, \frac{\sqrt{3}\,(2+k)+(2+\sqrt{2})\,|k|}{2+k} + \frac{2\sqrt{3}+3\,|k|}{2+k} \left(\frac{1}{h}+\frac{1}{h+1}\right)\right\},$$

$$(6.4)\qquad N_2 = \left\{\frac{10\sqrt{3}+(\sqrt{3}+14)\,|k|}{2\,(2+k)} \sup_{y\in A} \int_{\Sigma} \left|\frac{\partial}{\partial n^x}\, \frac{1}{|x-y|}\right| d\sigma_x + \frac{|k|}{2+k}\, \varepsilon^{-2}\,(\text{area}\ \Sigma)\right\} |a_n|,$$

$$(6.5)\qquad N_3 = 3\left(\sqrt{3}+|k|+\frac{\sqrt{2}\,|k|}{2+k}\right) F_5\, \varepsilon^{-3} \max_{y\in \bar{A}} \int_{\Sigma} \frac{d\sigma_x}{|x-y|}\,.$$

Si fissi l'intero n tale che, oltre le (2.14), sia anche verificata la diseguaglianza:

$$N_1\, |a_n| > 2\pi \left(1 - \frac{|k|}{2+k}\, b_n\right).$$

Il numero positivo ε sia tale che, oltre a soddisfare la (2.15), soddisfi anche la seguente condizione:

$$\varepsilon \leq \left[\frac{2\pi\left(1-\frac{|k|}{2+k}\, b_n\right)}{N_1\, |a_n|}\right]^{1/h} \varrho_A\,.$$

Dalla (6.2) si trae allora la (1.10) con

$$(6.6)\qquad H_8 = \frac{N_2}{2\pi\left(1-\frac{|k|}{2+k}\, b_n\right) - N_1\, |a_n|\, \varepsilon^h}\,; \qquad H_9 = \frac{N_3}{2\pi\left(1-\frac{|k|}{2+k}\, b_n\right) - N_1\, |a_n|\, \varepsilon^h}\,.$$

Si noti che se, scelto comunque un punto x^0 in A e una semiretta di origine in x^0, questa sega Σ in un numero finito di continui disgiunti che non supera mai il numero fisso μ, riesce allora

$$\sup_{y\in A} \int_{\Sigma} \left|\frac{\partial}{\partial n^x}\, \frac{1}{|x-y|}\right| d\sigma_x \leqq 4\pi\, \mu\,.$$[13]

Dalla (1.10) e dalla (1.9) segue la (1.2) con

$$(6.7)\qquad H = H_8 + H_7\, H_9\,.$$

Resta così provata la (2) per le soluzioni della (1) appartenenti a $C^1(\bar{A}) \cap C^2(A)$.

Da (3.6), (4.1), (6.7) si trae:

$$H = H_8 + H_1\, H_5\, H_9 + H_2\, H_3\, H_5\, H_9\, (\text{vol}\ A)^{\frac{1}{2}} + H_6\, H_9$$

provvedendo al calcolo esplicito delle costanti a secondo membro le (2.16), (3.5), (4.2), (4.3), (6.6).

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[13] *Cfr.* [9].

La costante H si può far quindi dipendere, oltre che dal parametro k, dalle seguenti *costanti geometriche* del campo A: c, h, ϱ_A, vol A, d, δ, μ.

7. Dimostrazione del punto VI

Supponiamo che Σ verifichi l'ipotesi, più particolare, di essere di classe C_h^2. Sia $\lambda(\xi)$ un versore definito in ogni punto ξ di Σ, le cui componenti siano funzioni di classe 2 su Σ e che formi col versore normale a Σ interno ad A un angolo $< \frac{1}{2}\pi$. L'esistenza di un versore quale $\lambda(\xi)$ è ovvia. Esiste un $\varrho_0 > 0$ tale che per $0 < \varrho \leq \varrho_0$ il luogo Σ_ϱ descritto dal punto $x = \xi + \varrho\lambda(\xi)$ quando ξ descrive Σ, è la frontiera di un campo A_ϱ, la cui chiusura è contenuta in A. Si ha, detto u un arbitrario vettore di $C^2(\bar{A})$ e v una soluzione della (1) appartenente a $C^0(\bar{A}) \cap C^2(A)$:

$$(7.1) \qquad -\int_{A_\varrho} v\, E(u)\, dx + \int_{\Sigma_\varrho} \{u\, L_\varrho(v) - v\, L_\varrho(u)\}\, d\sigma_\varrho = 0,$$

avendo indicato con L_ϱ l'operatore L calcolato su Σ_ϱ. Per la continuità di v in $\bar{A}$ riesce:

$$(7.2) \qquad \lim_{\varrho \to 0} \int_{A_\varrho} v\, E(u)\, dx = \int_A v\, E(u)\, dx; \qquad \lim_{\varrho \to 0} \int_{\Sigma_\varrho} v\, L_\varrho(u)\, d\sigma_\varrho = \int_\Sigma v\, L(u)\, d\sigma.$$

Diciamo $\bar{u}$ un arbitrario vettore di classe 2 su Σ e sia u tale da appartenere a $C^2(\bar{A})$ e da esser così definito in $\bar{A} - A_{\frac{1}{2}\varrho_0}$: $u[\xi + \varrho\lambda(\xi)] = \bar{u}(\xi)$. Indichiamo con $\mathscr{D}_2$ lo spazio dei vettori (a tre componenti) di classe due su Σ munito della ordinaria topologia della convergenza uniforme delle funzioni e delle derivate prime e seconde di queste (rispetto ai parametri locali su Σ). Sia $\mathscr{D}_2^*$ lo spazio duale di $\mathscr{D}_2$. Poniamo per $0 < \varrho < \frac{1}{2}\varrho_0$

$$\langle T_\varrho, \bar{u} \rangle = \int_{\Sigma_\varrho} u\, L_\varrho(v)\, d\sigma_\varrho = \int_\Sigma \bar{u}\, L_\varrho(v) \frac{d\sigma_\varrho}{d\sigma}\, d\sigma;$$

riesce $T_\varrho \in \mathscr{D}_2^*$. Qualunque sia $\bar{u}$, esiste, per le (7.1), (7.2), $\lim_{\varrho \to 0} \langle T_\varrho, \bar{u} \rangle$. Ne segue, per il ben noto principio di Banach-Steinhaus della uniforme limitatezza[14], l'esistenza di un elemento $T \in \mathscr{D}_2^*$ tale che $\lim_{\varrho \to 0} \langle T_\varrho, \bar{u} \rangle = \langle T, \bar{u} \rangle$. Segue allora dalla (7.1):

$$\int_A v\, E(u)\, dx + \int_\Sigma v\, L(u)\, d\sigma = \langle T, u \rangle$$

e, quindi, per $u \in \mathfrak{U}_0$, il sussistere della (1.4). Dalla proprietà di completezza stabilita per $\mathfrak{U}_0$ nel § 5, si deduce, per dualità dalla (1.2), scritta per $u \in \mathfrak{U}_0$, il sussistere della (2) per le soluzioni di (1) appertenenti a $C^0(\bar{A}) \cap C^2(A)$.

L'ipotesi, ammessa al principio di questo § 7, che Σ sia di classe C_h^2, si potrebbe rimuovere osservando che H dipende dalle costanti di Hölder degli elementi differenziali del primo ordine di Σ, epperò, supposto Σ soltanto di classe C_h^1, basterebbe approssimare uniformemente Σ con una successione Σ_n di frontiere di domini $\bar{A}_n$, interni ad A, tali che i coefficienti di Hölder c_n (relativi ad un fissato esponente h) degli elementi del primo ordine su Σ_n si mantengono equilimitati, riuscendo ogni Σ_n di classe C_h^2. Ma di ciò ci permettiamo lasciare la cura al lettore desideroso di tale estensione.

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[14] *Cfr.* [8].

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On the Solutions of a Class of Equations Occurring in Continuum Mechanics, with Application to the Stokes Paradox

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1. Introduction

In this paper we apply methods developed by Finn in [*1*] to study asymptotic properties of solutions of equations

$$(1.1\,\mathrm{a})\qquad \Delta \vec{\boldsymbol{w}} - \vec{\boldsymbol{b}} \cdot \nabla \vec{\boldsymbol{w}} - \nabla p = 0, \qquad \vec{\boldsymbol{b}} = \text{const.},$$

$$(1.1\,\mathrm{b})\qquad \nabla \cdot \vec{\boldsymbol{w}} = -c\,p, \qquad c = \text{const.},$$

for a vector field $\vec{\boldsymbol{w}}(\boldsymbol{x}) = \{w_1, \ldots, w_n\}$ and scalar field $p(\boldsymbol{x})$ which are defined in a neighborhood $\mathscr{E}$ of infinity in n-dimensional Euclidian space. The notation in (1.1) is the usual one of vector analysis. By choosing particular values of the constant c and the constant vector $\vec{\boldsymbol{b}}$, we obtain from (1.1) the following equations of fluid and solid mechanics (see, *e.g.*, [*2*, *3*]):

i)	$c = 0, \quad \vec{\boldsymbol{b}} = 0$	Stokes equations of hydrodynamics
ii)	$c = 0$	Oseen equations of hydrodynamics
iii)	$\vec{\boldsymbol{b}} = 0$	Equations of linear elasticity.

We assume in most of this paper, as is the case in the problems of mechanics from which the equations are derived, that $c \geqq 0$. However, in some contexts, notably the basic Theorems 1 and 4, it is sufficient to assume $c > -1$.

The main result of this study consists in showing that under a surprisingly weak growth condition at infinity, the solution admits a representation in terms of a single surface integral over an inner bounding surface Σ. From this representation we conclude that the solution tends to a limit $(\vec{\boldsymbol{w}}_0, p_0)$ at infinity, and the rate of decay of $(\vec{\boldsymbol{w}}, p)$ to this limit is controlled by the asymptotic properties of the fundamental solution tensor associated with (1.1). We determine this tensor explicitly in §3. In §4 we apply our result to a discussion of solutions for which $\vec{\boldsymbol{w}}(\boldsymbol{x})$ vanishes on Σ. These solutions are of importance physically as steady fluid flows or as deformations exterior to a rigid body. We obtain in particular a new clarification of the Stokes Paradox of Hydrodynamics, which states essentially that in two dimensions a solution of the (time-independent) Stokes equations in $\mathscr{E}$ which vanishes on Σ is necessarily unbounded. Our result is sharp in all dimensions and shows that the singular behavior of the solution is intimately connected with the forces exerted on Σ in the motion. We obtain as special cases all earlier results on this and on related problems which are known to us [*cf. 3*, p. 361, *4, 5, 6, 7*]; the demonstrations are in our opinion simpler and more natural than those previously employed.

The methods of this paper are not restricted to equations of the form (1.1), but are applicable to any elliptic system for which the fundamental solution tensor exhibits the qualitative properties needed in our demonstrations[1]. Equations with variable coefficients can also be treated, and it seems likely that a general description of asymptotic behavior can be constructed along lines suggested by N. G. MEYERS [*8*]. The particular choice of the equations (1.1) has been based on the consideration that they are general enough to indicate the scope of the methods, but not so general as to cause the ideas to be obscured by technical detail.

We remark that in the range of c considered the system (1.1) is elliptic $\left(\text{the characteristic determinant is } (1+c)\sum_{i=1}^{n}\xi_i^2>0 \text{ for } c>-1,\ \sum\xi_i^2>0\right)$ but not strongly elliptic, hence not of a type for which boundary value problems have been studied in recent literature. The properly set boundary value problem requires prescribing $\vec{\boldsymbol{w}}(\boldsymbol{x})$ on the boundary, but not the pressure $p(\boldsymbol{x})$, this function being determined by the solution[2]. Correspondingly, in the material

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[1] It would suffice, for example, for the fundamental solution tensor to have the property that for any prescribed positive integer k, all derivatives of sufficiently high order (depending on k) of each component of this tensor behave at infinity as $O(r^{-k})$. Professor L. HÖRMANDER has communicated to us a proof of the following equivalent condition, valid for any system with constant coefficients: Let $P(D)$, where $D=(-i\,D_1,\ldots,-i\,D_n)$, $i=$ imaginary unit, denote the determinant of the differential operators occurring in the system, and let, respectively, $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)$, $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_n)$ be vectors in real or complex Euclidean n-space. Then the fundamental solution tensor has the above stated property if and only if both the conditions

a) $P(\boldsymbol{\xi})\neq 0$ for $\boldsymbol{\xi}\neq 0$,

and

b) $\operatorname{Im}\boldsymbol{\zeta}\to\infty$ if $\boldsymbol{\zeta}\to\infty$ on the surface $P(\boldsymbol{\zeta})=0$,

are satisfied by the polynomial P. The representation theorems of this paper, and the corresponding asymptotic estimates of the solutions, can be extended without essential change to all systems which satisfy the conditions of HÖRMANDER.

Such systems include, for example, the linearized magnetohydrodynamic equations

$$\Delta\vec{\boldsymbol{w}}-M^2\left[\vec{\boldsymbol{w}}-\frac{(\vec{\boldsymbol{w}}\cdot\vec{\boldsymbol{w}}_0)}{|\vec{\boldsymbol{w}}_0|}\,\frac{\vec{\boldsymbol{w}}_0}{|\vec{\boldsymbol{w}}_0|}\right]=0,$$
$$\nabla\cdot\vec{\boldsymbol{w}}=0$$

for which the essential features of Theorem 1 are easily seen to apply. In this case, there is no Stokes Paradox, since the fundamental solution tensor vanishes at infinity for all $n\geqq 2$.

[2] The existence of a solution corresponding to such boundary data has to our knowledge been proved in the literature only in the special case $c=\vec{\boldsymbol{b}}=0$ (Stokes equations). It seems likely that, starting with the solution of ODQVIST [*9*] for this case, the existence in the general case can be demonstrated by the continuity method, using estimates of the type studied in [*10*] and in [*11*]. A direct demonstration by the method of surface potentials in the case $c=0$, $\vec{\boldsymbol{b}}$ arbitrary, has been given by J. SANDERS in a work to appear shortly.

of this paper no assumption is made on the behavior of the pressure, and the regularity of the pressure at infinity is proved as a consequence of growth assumptions on $\vec{w}(\boldsymbol{x})$ alone.

2. Notation and Definitions; the Integral Representation

The following notation will be used. We choose the origin O of our coordinate system to be a point interior to Σ. The set of all points exterior to Σ is denoted by $\mathscr{E}$. Let Σ_R denote a sphere of radius R centered at O and enclosing Σ. We denote by $\mathscr{I}_R$ the annular region bounded between Σ and Σ_R. The region exterior to Σ_R is denoted by $\mathscr{E}_R$.

Let $\vec{w}(\boldsymbol{x})=[w_i(\boldsymbol{x})]$ and $T(\boldsymbol{x})=[T_{ij}(\boldsymbol{x})]$ be, respectively, vector and tensor fields defined in $\mathscr{E}$. We define

$$|\vec{w}(\boldsymbol{x})|=\left[\sum_{i=1}^{n} w_i^2(\boldsymbol{x})\right]^{\frac{1}{2}},$$

$$|T(\boldsymbol{x})|=\left[\sum_{i,j=1}^{n} T_{ij}^2(\boldsymbol{x})\right]^{\frac{1}{2}}.$$

We write $T_{ij}=O(R^{\alpha})$ if there exists a constant A so that $|T_{ij}(\boldsymbol{x})|\leq A|\boldsymbol{x}|^{\alpha}$ as $\boldsymbol{x}\to\infty$; we write $T_{ij}=o(R^{\alpha})$ if $|\boldsymbol{x}|^{-\alpha}|T_{ij}(\boldsymbol{x})|\to 0$ as $\boldsymbol{x}\to\infty$. Similar definitions apply to vector and higher-order tensor fields.

Corresponding to a given vector field $\vec{u}(\boldsymbol{x})$ and scalar $p(\boldsymbol{x})$, we define the second order tensor field

$$T\vec{u}=\{T\vec{u}\}_{ij}=-(1-c)\,p\,\delta_{ij}+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{2.1}$$

where

$$\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

We also define the linear differential operators $L\vec{u}$ and $M\vec{v}$ by the expressions

$$L\vec{u}=\Delta\vec{u}-\nabla p-\vec{b}\cdot\nabla\vec{u},$$

$$M\vec{v}=\Delta\vec{v}-\nabla q+\vec{b}\cdot\nabla\vec{v}.$$

For any vector fields $\vec{u}$ and $\vec{v}$ and corresponding scalars p, q satisfying the conditions

$$\nabla\cdot\vec{u}=-c\,p,\qquad \nabla\cdot\vec{v}=-c\,q \tag{2.2}$$

in a region V with sufficiently smooth boundary S, we have always the identity

$$\int_V(\vec{v}\cdot L\vec{u}-\vec{u}\cdot M\vec{v})\,dV=\int_S[\vec{v}\cdot T\vec{u}-\vec{u}\cdot T\vec{v}-(\vec{u}\cdot\vec{v})(\vec{b}\cdot\vec{n})]\,dS \tag{2.3}$$

where by $\vec{v}\cdot T\vec{u}$ is understood the expression

$$\vec{v}\cdot T\vec{u}=v_i\{T\vec{u}\}_{ij}\,n_j,$$

$\vec{n}=\{n_j\}$ is the unit exterior directed normal on S, and summation is extended over repeated indices.

Let $\boldsymbol{\chi}(\boldsymbol{x}-\boldsymbol{y})=\{\chi_{ij}\}$, $\boldsymbol{\psi}(\boldsymbol{x}-\boldsymbol{y})=\{\psi_i\}$ denote an $n\times n$ matrix and an n-vector, respectively, which become singular at $\boldsymbol{x}=\boldsymbol{y}$ in such a way that

$$\delta_{ik}=\lim_{r\to 0}\int_{\Sigma_r}[T\boldsymbol{\chi}+\boldsymbol{\chi}(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{n}})]\,dS$$

where Σ_r denotes the surface of a sphere of radius r about $\boldsymbol{x}$ as center and $T\boldsymbol{\chi}$ is formed by interpreting the components of $\boldsymbol{\psi}$ as pressures. We obtain from (2.3) the integral representation

$$\vec{\boldsymbol{u}}(\boldsymbol{x})=\int_V(\boldsymbol{\chi}\cdot L\vec{\boldsymbol{u}}-\vec{\boldsymbol{u}}\cdot M\boldsymbol{\chi})\,dV-\int_S[\boldsymbol{\chi}\cdot T\vec{\boldsymbol{u}}-\vec{\boldsymbol{u}}\cdot T\boldsymbol{\chi}-(\vec{\boldsymbol{u}}\cdot\boldsymbol{\chi})(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{n}})]\,dS. \tag{2.4}$$

In (2.4), $\boldsymbol{x}$ appears as a parameter, the differentiation and integration being performed with respect to $\boldsymbol{y}$.

Let now $\vec{\boldsymbol{w}}(\boldsymbol{x})$ and $\boldsymbol{\chi}(\boldsymbol{x}-\boldsymbol{y})$ be vector and tensor fields which satisfy (2.2) and also the equations

$$L(\vec{\boldsymbol{w}})=0,\qquad M(\boldsymbol{\chi})=0$$

for each column vector in the matrix (χ_{ij}), with $\boldsymbol{y}$ as independent variable. We have then the integral representation for the solutions of (1.1),

$$\vec{\boldsymbol{w}}(\boldsymbol{x})=-\int_S[\boldsymbol{\chi}\cdot T\vec{\boldsymbol{w}}-\vec{\boldsymbol{w}}\cdot T\boldsymbol{\chi}-(\vec{\boldsymbol{w}}\cdot\boldsymbol{\chi})(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{n}})]\,dS. \tag{2.5}$$

A similar reasoning yields the representation for the scalar $p(\boldsymbol{x})$,

$$p(\boldsymbol{x})=-\int_S[\boldsymbol{\psi}\cdot T\vec{\boldsymbol{w}}-\vec{\boldsymbol{w}}\cdot T\boldsymbol{\psi}-(\vec{\boldsymbol{w}}\cdot\boldsymbol{\psi})(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{n}})]\,dS. \tag{2.6}$$

In forming $T\boldsymbol{\psi}$ we have introduced a "pressure", $\psi^*=\vec{\boldsymbol{b}}\cdot\nabla\left(\frac{1}{r}\right)$.

3. The Fundamental Solution

The fundamental solution $(\boldsymbol{\chi},\boldsymbol{\psi})$ is determined from the equations

$$\frac{\partial\chi_{ki}}{\partial y_i}=-c\,\psi_k, \tag{3.1 a}$$

$$\Delta\chi_{ki}-\frac{\partial\psi_k}{\partial y_i}+\vec{\boldsymbol{b}}\cdot\nabla\chi_{ki}=\delta_{ki}\,\delta(\boldsymbol{y}-\boldsymbol{x}). \tag{3.1 b}$$

One puts (*cf.* [*12*, p. 31])

$$\psi_k=-\frac{\partial}{\partial y_k}(\Delta+\vec{\boldsymbol{b}}\cdot\nabla)\,\Phi, \tag{3.2a}$$

$$\chi_{ki}=\delta_{ki}[(1+c)\,\Delta+c\,\vec{\boldsymbol{b}}\cdot\nabla]\,\Phi-\frac{\partial^2}{\partial y_i\,\partial y_k}\Phi. \tag{3.2b}$$

Equation (3.1 a) is then identically satisfied, while equation (3.1 b) reduces to

$$(\Delta+\vec{\boldsymbol{b}}\cdot\nabla)\,[(1+c)\,\Delta+c\,\vec{\boldsymbol{b}}\cdot\nabla]\,\Phi=\delta(\boldsymbol{y}-\boldsymbol{x}). \tag{3.3}$$

If $\vec{\boldsymbol{b}}\neq 0$, let $b=|\vec{\boldsymbol{b}}|$, $\tau=\frac{\vec{\boldsymbol{b}}\cdot(\boldsymbol{y}-\boldsymbol{x})}{b}$. A solution of (3.3) can be determined in the form

$$\Phi(\boldsymbol{y}-\boldsymbol{x})=\frac{1}{b}\int^{\tau}[(1+c)\,\Phi_2-\Phi_1]\,d\tau \tag{3.4}$$

where Φ_1 and Φ_2 are solutions of the equations

$$(3.5)\qquad (\Delta + \vec{\boldsymbol{b}}\cdot\nabla)\,\Phi_1 = \delta(\boldsymbol{y}-\boldsymbol{x});\qquad [(1+c)\,\Delta + c\,\vec{\boldsymbol{b}}\cdot\nabla]\,\Phi_2 = \delta(\boldsymbol{y}-\boldsymbol{x}).$$

If $c \neq 0$,

$$(3.6\text{a})\qquad \Phi_1 = -\frac{1}{2\pi}\left(\frac{b}{4\pi r}\right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}\left(\frac{b r}{2}\right) e^{-\frac{1}{2}\vec{\boldsymbol{b}}\cdot(\boldsymbol{y}-\boldsymbol{x})},$$

$$(3.6\text{b})\qquad \Phi_2 = -\frac{1}{2\pi(1+c)}\left(\frac{b|c|}{4\pi|1+c|\,r}\right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}\left(\frac{b|c|}{2|1+c|}\,r\right) e^{-\frac{c}{2(1+c)}\vec{\boldsymbol{b}}\cdot(\boldsymbol{y}-\boldsymbol{x})},$$

while if $c=0$, the expression for Φ_2 is replaced by

$$(3.6\text{c})\qquad \Phi_2 = \frac{1}{2\pi}\log r, \qquad n=2,$$

$$(3.6\text{d})\qquad \Phi_2 = -\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}(n-2)\,r^{n-2}}, \qquad n \geqq 3,$$

where $r = |\boldsymbol{y}-\boldsymbol{x}|$ and $K_{\frac{n-2}{2}}$ is a modified Bessel function of the second kind (*cf.* [*12*, p. 36]); if $\vec{\boldsymbol{b}}=0$, equation (3.3) reduces to

$$(3.7\text{a})\qquad (1+c)\,\Delta\Delta\Phi = \delta(\boldsymbol{y}-\boldsymbol{x}),$$

and we may take

$$(3.7\text{b})\qquad \Phi = \frac{(-1)^{\frac{n-2}{2}}\, r^{4-n}\log r}{8\pi^{\frac{n}{2}}\left(2-\frac{n}{2}\right)!\,(1+c)}, \qquad n = 2 \text{ and } 4$$

and

$$(3.7\text{c})\qquad \Phi = \frac{\Gamma\left(\frac{n}{2}-2\right) r^{4-n}}{16\pi^{\frac{n}{2}}(1+c)}, \qquad n = \text{odd and } n = \text{even} > 4.$$

When these expressions are substituted into (3.4) and (3.2), the fundamental solution tensor (χ, Ψ) is obtained explicitly. From (3.6) and (3.7) we may compute formally all asymptotic properties of this tensor. However, for the principal purposes of this paper we need only the following crude estimates as $r \to \infty$, easily obtained from the form of the expression[3]:

a) $\vec{\boldsymbol{b}} = 0$:

$$(3.8\text{a})\qquad |\boldsymbol{\chi}| = O(\log r), \qquad |\boldsymbol{\Psi}| = O(r^{-1}), \qquad n=2,$$

$$(3.8\text{b})\qquad |\boldsymbol{\chi}| = O(r^{2-n}), \qquad |\boldsymbol{\Psi}| = O(r^{1-n}), \qquad n>2,$$

$$(3.8\text{c})\qquad |\nabla^{(k)}\boldsymbol{\chi}| = O(r^{2-n-k}), \qquad |\nabla^{(k)}\boldsymbol{\Psi}| = O(r^{1-n-k}), \qquad k \geqq 1, \quad n \geqq 2$$

where $\nabla^{(k)}$ denotes a k^{th} iterate of the gradient operator.

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[3] Somewhat more detailed estimates in the case $c=0$ are given in [*11*] and [*13*].

b) $\vec{\boldsymbol{b}} \neq 0$:

$$|\boldsymbol{\chi}| = O(r^{-\frac{1}{2}}), \quad |\boldsymbol{\Psi}| = O(r^{-1}), \quad n = 2, \tag{3.9a}$$

$$|\nabla^{(k)} \boldsymbol{\chi}| = O\left(r^{-\frac{1+k}{2}}\right), \quad |\nabla^{(k)} \boldsymbol{\Psi}| = O\left(r^{-\frac{2+k}{2}}\right), \quad n = 2, \tag{3.9b}$$

$$|\boldsymbol{\chi}| = O\left(r^{-\frac{n-1}{2}}\right), \quad |\boldsymbol{\Psi}| = O\left(r^{-\frac{n}{2}}\right), \quad n \geqq 3, \tag{3.9c}$$

$$|\nabla^{(k)} \boldsymbol{\chi}| = O\left(r^{-\frac{n-1+k}{2}}\right), \quad |\nabla^{(k)} \boldsymbol{\Psi}| = O\left(r^{-\frac{n+k}{2}}\right), \quad k \geqq 1, \quad n \geqq 3. \tag{3.9d}$$

4. Statement of Results

In this section we state the main theorems of our paper. We prove these theorems in §6. Because of the qualitative difference in asymptotic behavior which occurs, we discuss separately the case $\vec{\boldsymbol{b}} = 0$. It is the singular behavior of the solutions in this case which gives rise to the Stokes Paradox. The case $\vec{\boldsymbol{b}} \neq 0$, and some theorems valid for every choice of $\vec{\boldsymbol{b}}$, are introduced later.

Case 1. $\vec{\boldsymbol{b}} = 0$.

Theorem 1 (Representation theorem). *Let $\vec{\boldsymbol{w}}(\boldsymbol{x})$ be a vector field defined in an n-dimensional neighborhood $\mathscr{E}$ of infinity and which satisfies* (1.1) *for some scalar field $p(\boldsymbol{x})$. Suppose that as $r \to \infty$, either*

i) $\vec{\boldsymbol{w}}(\boldsymbol{x}) = o(r)$

or

ii) $\int_{\mathscr{I}_R} |\vec{\boldsymbol{w}}|^2 r^{-(n+2)} dV = o(\log R)$

or

iii) $\int_{\mathscr{I}_R} |\nabla \vec{\boldsymbol{w}}|^2 r^{-n} dV = o(\log R)$.

Then there exist vector and scalar constants $\vec{\boldsymbol{w}}_0$ and p_0 such that throughout $\mathscr{E}$

$$\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 - \int_{\Sigma} [\boldsymbol{\chi} \cdot T \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \cdot T \boldsymbol{\chi}] \, dS, \tag{4.1a}$$

$$p(\boldsymbol{x}) = p_0 - \int_{\Sigma} [\boldsymbol{\psi} \cdot T \vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \cdot T \boldsymbol{\psi}] \, dS. \tag{4.1b}$$

The fields $\vec{\boldsymbol{w}}(\boldsymbol{x})$ and $p(\boldsymbol{x})$ admit the asymptotic expressions

$$\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + \boldsymbol{\chi}(\boldsymbol{x}) \cdot \vec{\boldsymbol{\mathfrak{T}}} + O(r^{1-n}), \tag{4.2a}$$

$$p(\boldsymbol{x}) = p_0 + \boldsymbol{\psi}(\boldsymbol{x}) \cdot \vec{\boldsymbol{\mathfrak{T}}} + O(r^{-n}) \tag{4.2b}$$

where

$$\vec{\boldsymbol{\mathfrak{T}}} = -\int_{\Sigma} T \vec{\boldsymbol{w}} \, dS.$$

Theorem 1 is best possible, in the sense that "o" cannot be replaced by "O" in the hypotheses. We remark that the vector $\vec{\boldsymbol{\mathfrak{T}}}$ can be interpreted physically as the force exerted on the contour Σ in the motion or displacement.

A particular consequence of Theorem 1 is that

$$|\vec{w}(x) - \vec{w}_0| = \begin{cases} O(\log r), & n = 2, \\ O(r^{2-n}), & n > 2 \end{cases}$$

with an improved behavior at infinity in case $\vec{\mathfrak{T}} = 0$. This observation bears on our next theorem, in which we introduce a boundary condition on Σ.

Theorem 2. *Under the hypotheses of Theorem* 1, *if* $\vec{w}(x) = 0$ *on* Σ, *then* $\vec{\mathfrak{T}} \neq 0$ *unless* $\vec{w}(x) \equiv 0$ *in* $\mathscr{E}$. *If* $n \geqq 3$, *then*

$$\vec{\mathfrak{T}} \cdot \vec{w}_0 = 2 \int_{\mathscr{E}} (\operatorname{def} \vec{w})^2 \, dV + c(1-c) \int_{\mathscr{E}} p^2 \, dV \tag{4.3}$$

where def $\vec{w}$ *denotes the matrix,* $\frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$.

The physical content of (4.3) is that in any steady fluid motion past Σ with adherence on the boundary, the work done at the boundary per unit time equals the rate at which energy is converted into heat in the motion. Thus, in three or more dimensions no energy is introduced into the flow at infinity. This is not the case in two dimensions when $\vec{b} = 0$. In the interpretation of $\vec{w}(x)$ as an elastic deformation, (4.3) asserts that the work done at the boundary due to a constant deformation $-\vec{w}_0$ on Σ equals the potential energy of deformation in the medium. A particular consequence of (4.3) is that in any fluid flow past a rigid body Σ, or in any elastic deformation of a medium which surrounds Σ and is stationary at infinity, the body necessarily experiences a "drag" force in the direction of $\vec{w}_0$.

Theorem 3. *Let* $\vec{w}(x)$ *be a solution of* (1.1) *in* $\mathscr{E}$. *If for some* $\vec{w}_0$,

$$\vec{w}(x) - \vec{w}_0 = \begin{cases} o(\log r), & n = 2, \\ o(r^{2-n}), & n > 2, \end{cases}$$

then $\vec{\mathfrak{T}} = 0$.

From Theorems 2 and 3 we obtain in particular the Stokes Paradox of Hydrodynamics. For if $\vec{w}(x)$ represents a two-dimensional fluid motion exterior to Σ, and if $\vec{w}(x)$ is bounded, then necessarily $\vec{\mathfrak{T}} = 0$. But if the fluid adheres to Σ, then $\vec{w}(x) = 0$ on Σ, hence $\vec{\mathfrak{T}} \neq 0$, a contradiction. If $c = 0$ our result is a strengthened form of the results in [*3, 4, 5, 6*]; if $c > 0$, we obtain a result which includes the extension to this case by Duffin & Noll [*7*].

Case 2. $\vec{b} \neq 0$.

Theorem 4. *Under the hypotheses of Theorem* 1, *there exist vector and scalar constants* $\vec{w}_0$ *and* p_0 *such that throughout* $\mathscr{E}$,

$$\vec{w}(x) = \vec{w}_0 - \int_{\Sigma} [\chi \cdot T\vec{w} - \vec{w} \cdot T\chi - (\vec{w} \cdot \chi)(\vec{b} \cdot \vec{n}) \, dS, \tag{4.4a}$$

$$p(x) = p_0 - \int_{\Sigma} [\psi \cdot T\vec{w} - \vec{w} \cdot T\psi - (\vec{w} \cdot \psi)(\vec{b} \cdot \vec{n})] \, dS. \tag{4.4b}$$

The fields $\vec{\boldsymbol{w}}(\boldsymbol{x})$ *and* $p(\boldsymbol{x})$ *admit the asymptotic expressions*

$$\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + \boldsymbol{\chi}\cdot\vec{\mathfrak{M}} + O(r^{-1}), \qquad n=2, \tag{4.5 a}$$

$$\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + \boldsymbol{\chi}\cdot\vec{\mathfrak{M}} + O\left(r^{-\frac{n}{2}}\right), \qquad n>2, \tag{4.5 b}$$

$$p(\boldsymbol{x}) = p_0 + \boldsymbol{\psi}\cdot\vec{\mathfrak{M}}^* + O\left(r^{-\frac{n+1}{2}}\right), \tag{4.5 c}$$

where

$$\mathfrak{M} = -\int_{\Sigma} [T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{n}})]\, dS$$

and

$$\mathfrak{M}^* = -\int_{\Sigma} [T\vec{\boldsymbol{w}} - \vec{\boldsymbol{b}}(\vec{\boldsymbol{w}}\cdot\vec{\boldsymbol{n}}) - \vec{\boldsymbol{w}}(\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{n}})]\, dS.$$

Theorem 5. *Under the hypotheses of Theorem* 1, *if* $\vec{\boldsymbol{w}}(\boldsymbol{x}) = 0$ *on* Σ, *then* $\vec{\mathfrak{T}} \neq 0$ *unless* $\vec{\boldsymbol{w}}(\boldsymbol{x}) \equiv 0$ *in* $\mathscr{E}$. *Also, for all* $n \geqq 2$,

$$\vec{\mathfrak{T}}\cdot\vec{\boldsymbol{w}}_0 = 2\int_{\mathscr{E}} (\operatorname{def}\vec{\boldsymbol{w}})^2\, dV + c(1-c)\int_{\mathscr{E}} p^2\, dV. \tag{4.6}$$

The following theorems are valid for every choice of $\vec{\boldsymbol{b}}$. The first of them includes the work of Finn & Noll [*6*] and of Duffin & Noll [*7*] on the same question.

Theorem 6 (Uniqueness theorem). *Let* $\vec{\boldsymbol{w}}(\boldsymbol{x})$ *be a solution of* (1.1) *in* $\mathscr{E}$ *for some scalar field* $p(\boldsymbol{x})$. *Let* $\vec{\boldsymbol{w}}(\boldsymbol{x}) = 0$ *on* Σ *and suppose that as* $\boldsymbol{x}\to\infty$ *either*

i) $\vec{\boldsymbol{w}}(\boldsymbol{x}) \to 0$

or

ii) $\int_{\mathscr{S}_R} |\vec{\boldsymbol{w}}|^2 r^{-n}\, dV = o(\log R)$.

Then $\vec{\boldsymbol{w}}(\boldsymbol{x}) \equiv 0$ *in* $\mathscr{E}$.

With any fluid motion for which $\vec{\boldsymbol{w}}(\boldsymbol{x}) \to \vec{\boldsymbol{w}}_0$ at infinity there is associated the kinetic energy of disturbance from the uniform flow, $E = \int_{\mathscr{E}} |\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}_0|^2 dV$. In any potential fluid motion, this quantity is always finite and can be used as the basis for a variational approach to the study of such motions. The situation is, however, quite different in the case of solutions of (1.1), as can be seen from the following theorem[4]:

Theorem 7. *Let* $n=2$ *or* 3, *and let* $\vec{\boldsymbol{w}}(\boldsymbol{x})$ *be a solution of* (1.1) *which satisfies the hypotheses of Theorem* 1 *and for which* $\vec{\boldsymbol{w}}(\boldsymbol{x}) = 0$ *on* Σ, $\vec{\boldsymbol{w}}(\boldsymbol{x}) \not\equiv 0$ *in* $\mathscr{E}$. *Then* $\int_{\mathscr{E}} |\vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0|^2 dV = \infty$ *for every choice of* $\vec{\boldsymbol{w}}_0$.

5. Preliminary Lemmas

Lemma 1. *Let* $u(\boldsymbol{x})$ *be any Cartesian component of a vector field* $\vec{\boldsymbol{w}}(\boldsymbol{x})$ *which satisfies* (1.1) *in* $\mathscr{E}$ *for some scalar field* $p(\boldsymbol{x})$. *Suppose that for all sufficiently large* R,

$$\int_{\mathscr{S}_{R+1}} u^2 r^{-s}\, dV < K(R) \tag{5.1}$$

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[4] A theorem of this type has been proved also for the (non-linear) Navier-Stokes hydrodynamical equations, in two dimensions by W. Wolibner [*14*], and for the three-dimensional case by Finn [*15*]. Important extensions of Wolibner's results to time-dependent motions of compressible fluids in two and in three dimensions have been given by A. Krzywicki [*16*].

for some real s. Then for any non-negative integer k,

$$|\nabla^{(k)} u(\boldsymbol{x})| = O[r^{s/2} K^{\frac{1}{2}}(r)]$$

where $\nabla^{(k)}$ *denotes the* k^{th} *iterative of the gradient operator.*

Proof (*cf.* [*17*]). One finds from (1.1) the following equation governing $u(\boldsymbol{x})$:

$$L_1 u = \{(1+c)\Delta - c\,\vec{\boldsymbol{b}}\cdot\nabla\}\{\Delta - \vec{\boldsymbol{b}}\cdot\nabla\}u = 0.$$

If one defines the adjoint operator L_2 by

$$L_2 = \{(1+c)\Delta + c\,\vec{\boldsymbol{b}}\cdot\nabla\}\{\Delta + \vec{\boldsymbol{b}}\cdot\nabla\},$$

one obtains the following identity for any scalar functions φ and ψ in a region $\mathscr{I}$:

$$\varphi L_1\psi - \psi L_2\varphi = \operatorname{div}\vec{\boldsymbol{W}} \tag{5.2}$$

where

$$\begin{aligned}\vec{\boldsymbol{W}} = {} & (1+c)\,[\varphi\nabla(\Delta\psi) - \Delta\psi\nabla\varphi + \Delta\varphi\nabla\psi - \psi\nabla(\Delta\varphi)] - \\ & - (1+2c)\,[\varphi\nabla(\vec{\boldsymbol{b}}\cdot\nabla\psi) + \psi\nabla(\vec{\boldsymbol{b}}\cdot\nabla\varphi) - \vec{\boldsymbol{b}}(\nabla\varphi\cdot\nabla\psi)] + \\ & + c\,\vec{\boldsymbol{b}}\,[\varphi\,\vec{\boldsymbol{b}}\cdot\nabla\psi - \psi\,\vec{\boldsymbol{b}}\cdot\nabla\varphi].\end{aligned}$$

Let P be an arbitrary point of large magnitude in $\mathscr{E}$. Construct a unit sphere S with P as center and let Q be an arbitrary point of S. Let $\zeta(Q)$ be an infinitely differentiable function defined in S such that

$$\zeta(Q) = \begin{cases} 1 & \text{for} \quad |Q-P| \leqq \frac{1}{4}, \\ 0 & \text{for} \quad |Q-P| \geqq \frac{3}{4}. \end{cases}$$

Let us now put $\psi = u$ and $\varphi = \zeta\Phi$ in (5.2); u and Φ being the solutions of

$$L_1 u = 0, \qquad L_2\Phi = \delta(P-Q).$$

Integrating (5.2) over the unit sphere S, we find

$$u(P) = -\int_{\mathscr{A}} u(Q)\,L_2[\zeta(Q)\,\Phi(P-Q)]\,dV_Q \tag{5.3}$$

where $\mathscr{A}$ is the annular region $\frac{1}{4} \leqq |P-Q| \leqq \frac{3}{4}$. We can differentiate the right-hand side of (5.3) with respect to P, thereby obtaining a representation for the derivatives of $u(P)$:

$$\nabla^{(k)} u(P) = -\int_{\mathscr{A}} u(Q)\,L_2[\zeta(Q)\,\nabla_P^{(k)}\Phi(P-Q)]\,dV_Q.$$

Thus, letting r denote distance to the origin of a reference system,

$$|\nabla^{(k)} u(P)|^2 \leqq \int_{\mathscr{A}} u^2 r^{-s}\,dV \int_{\mathscr{A}} r^s |L_2[\zeta(Q)\,\nabla_P^{(k)}\Phi(P-Q)|^2\,dV_Q$$

by Schwarz' Inequality. Thus for a suitable constant C,

$$\begin{aligned}|\nabla^{(k)} u(P)|^2 & \leqq C\,(r_P+1)^s \int_{\mathscr{I}_{R+1}} u^2 r^{-s}\,dV \\ & = O[r^s K(r)], \qquad \text{q.e.d.}\end{aligned}$$

Lemma 2. *Let $P(\boldsymbol{x})$ be a polynomial of degree m in $\mathscr{E}$, and let s be a real number such that*

$$\int_{\mathscr{I}_R} P^2(\boldsymbol{x})\, r^{-s}\, dV = o(\log R).$$

Then $P(\boldsymbol{x}) \equiv 0$ if $s \leqq n$, and $m < \frac{1}{2}(s-n)$ if $s > n$.

Proof. We have

$$P(\boldsymbol{x}) = a_m(\Omega)\, r^m + \cdots + a_0,$$

where the coefficients depend only on the position Ω on a unit sphere. We obtain

$$\begin{aligned}\int_{\mathscr{I}_R} P^2(\boldsymbol{x})\, r^{-s}\, dV &= \int_{\mathscr{I}_R} a_m^2\, r^{2m-s}\, r^{n-1}\, dr\, d\Omega + O\Big[\int_{\mathscr{I}_R} r^{2m-s-1}\, r^{n-1}\, dr\, d\Omega\Big] \\ &= k^2 \int^{r} r^{2m-s+n-1}\, dr + O(1) + o\left[r^{2m-s+n}\right]\end{aligned}$$

for some positive constant k^2.

Suppose $s \leqq n$. If then $m \geqq 0$, we shall have $\int_{\mathscr{I}_R} P^2(\boldsymbol{x})\, r^{-s} dV > k^2 \log R + O(1)$, a contradiction. Thus in this case $P(\boldsymbol{x}) \equiv 0$. If on the other hand $s > n$, then in order that $\int^{R} r^{2m-s+n-1} dr = o(\log R)$ we must have $2m < s-n$, q.e.d.

Lemma 3. *Let $\vec{\boldsymbol{w}}(\boldsymbol{x})$ be a continuously differentiable vector-valued function on and exterior to a unit n-sphere Σ, and let $\vec{\boldsymbol{w}}(\boldsymbol{x}) = 0$ on Σ. Then*

$$\int_{\mathscr{I}_R} |\vec{\boldsymbol{w}}|^2\, r^{-(n+2)}\, dV \leqq \int_{\mathscr{I}_R} |\nabla \vec{\boldsymbol{w}}|^2\, r^{-n}\, dV.$$

Proof. On an arbitrary ray through the center of Σ, we have

$$\begin{aligned}\int_1^R |\vec{\boldsymbol{w}}|^2\, r^{-(n+2)}\, r^{n-1}\, dr &= -\left.\frac{|\vec{\boldsymbol{w}}|^2}{2r^2}\right|_1^R + \int_1^R (\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}_s)\, r^{-2}\, dr \\ &\leqq \sqrt{\int_1^R |\vec{\boldsymbol{w}}|^2\, r^{-3}\, dr}\, \sqrt{\int_1^R |\vec{\boldsymbol{w}}_s|^2\, r^{-1}\, dr}\end{aligned}$$

so that

$$\int_1^R |\vec{\boldsymbol{w}}|^2\, r^{-(n+2)}\, r^{n-1}\, dr \leqq \int_1^R |\nabla \vec{\boldsymbol{w}}|^2\, r^{-n}\, r^{n-1}\, dr,$$

and the result then follows by an integration over concentric spheres.

6. Proofs of Theorems

Proof of Theorem 1. By (2.5) $\vec{\boldsymbol{w}}(\boldsymbol{x})$ has the representation $\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_1(\boldsymbol{x}) + \vec{\boldsymbol{w}}_2(\boldsymbol{x})$ where

$$\vec{\boldsymbol{w}}_1(\boldsymbol{x}) = -\int_{\Sigma} [\boldsymbol{\chi} \cdot T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \cdot T\boldsymbol{\chi}]\, dS, \tag{6.1}$$

$$\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = -\int_{\Sigma_R} [\boldsymbol{\chi} \cdot T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}} \cdot T\boldsymbol{\chi}]\, dS. \tag{6.2}$$

We show first that $\vec{\boldsymbol{w}}_1(\boldsymbol{x}) = \boldsymbol{\chi} \cdot \vec{\mathfrak{T}} + O(r^{1-n})$. We have, in fact,

$$\begin{aligned}\vec{\boldsymbol{w}}_1(x) &= -\boldsymbol{\chi}(\boldsymbol{x}) \cdot \int_{\Sigma} T\vec{\boldsymbol{w}}(\boldsymbol{y})\, dS_{\boldsymbol{y}} - \int_{\Sigma} \{[\boldsymbol{\chi}(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{\chi}(\boldsymbol{x})]\, T\vec{\boldsymbol{w}}(\boldsymbol{y}) - \vec{\boldsymbol{w}}(\boldsymbol{y})\, T\boldsymbol{\chi}(\boldsymbol{x}-\boldsymbol{y})\}\, dS_{\boldsymbol{y}} \\ &= \boldsymbol{\chi}(\boldsymbol{x}) \cdot \vec{\mathfrak{T}} + O(r^{1-n})\end{aligned}$$

since by the estimates (3.8c)

$$|T\chi| = O(r^{1-n}) \quad \text{for} \quad \boldsymbol{y} \text{ on } \Sigma,$$

and

$$|\chi(\boldsymbol{x}-\boldsymbol{y}) - \chi(\boldsymbol{x})| = |\boldsymbol{y}\cdot\nabla\chi(\boldsymbol{\xi})|$$

where

$$\boldsymbol{\xi} = \boldsymbol{x} - \alpha\boldsymbol{y}, \qquad 0 \leqq \alpha \leqq 1,$$

so that

$$|\chi(\boldsymbol{x}-\boldsymbol{y}) - \chi(\boldsymbol{x})| = O(r^{1-n})$$

again by the estimates (3.8c).

To estimate $\vec{\boldsymbol{w}}_2(\boldsymbol{x})$, we observe first that by Lemmas 1 and 3,

$$|\nabla^{(k)}\vec{\boldsymbol{w}}| = O(r^{s/2})$$

under any of the hypotheses of Theorem 1 for any $s > n+2$. (In applying Lemma 3, we may clearly assume that $\vec{\boldsymbol{w}}(\boldsymbol{x})$ vanishes in a suitable sphere.) From the equation (1.1) we then find $|\nabla p| = |\Delta\vec{\boldsymbol{w}}| = O(r^{s/2})$ for any such s, so that $|p(\boldsymbol{x})| = O(r^{(s+2)/2})$. Now

$$\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = -\int_{\Sigma_R} [\chi\cdot T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}\cdot T\chi]\, dS \tag{6.3}$$

for any sufficiently large R. Since the left-hand side in (6.3) is independent of R, so is the right-hand side. Similarly, we find

$$\nabla^{(k)}\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = -\int_{\Sigma_R} [\nabla^{(k)}\chi\cdot T\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}\cdot T\nabla^{(k)}\chi]\, dS$$

for any integer k, the integral being independent of R. But by the above estimates and the estimates (3.8c), we see that for any $k > \frac{s+4}{2}$, the integral over Σ_R tends to zero as $R\to\infty$. It follows that for any such k, $\nabla^{(k)}\vec{\boldsymbol{w}}_2(\boldsymbol{x}) \equiv 0$, from which we find that $\vec{\boldsymbol{w}}_2(\boldsymbol{x})$ is a polynomial of degree at most $k+1$, $\vec{\boldsymbol{w}}_2(\boldsymbol{x}) = P_m(\boldsymbol{x})$, $m \leqq k+1$. Thus, $\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_1(\boldsymbol{x}) + \vec{\boldsymbol{w}}_2(\boldsymbol{x}) = P_m(\boldsymbol{x}) + o(r)$ by the estimates (3.8a, b). If $\vec{\boldsymbol{w}}(\boldsymbol{x}) = o(r)$, we conclude immediately $P_m(\boldsymbol{x}) \equiv \text{const.} = \vec{\boldsymbol{w}}_0$, from which the stated representations of $\vec{\boldsymbol{w}}(\boldsymbol{x})$ follow. If

$$\int_{\mathscr{I}_R} |\vec{\boldsymbol{w}}|^2 r^{-(n+2)}\, dV = o(\log R),$$

we observe (since $|P_m(\boldsymbol{x})|^2 > c r^2$ in a semi-infinite cone with vertex at the origin) that necessarily

$$\int_{\mathscr{I}_R} |P_m(\boldsymbol{x})|^2 r^{-(n+2)}\, dV = o(\log R).$$

By Lemma 2, we have $m < \frac{n+2-n}{2} = 1$, i.e. $P_m(\boldsymbol{x}) \equiv \text{const.} = \vec{\boldsymbol{w}}_0$. Finally, if $\int_{\mathscr{I}_R} |\nabla\vec{\boldsymbol{w}}|^2 r^{-n}\, dV = o(\log R)$, the result follows from the above discussion by use of Lemma 3. The stated representations for $p(\boldsymbol{x})$ are obtained similarly.

Proof of Theorem 2. Let us suppose $\vec{\mathfrak{T}} = 0$. Then by Theorem 1, $\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + O(r^{1-n})$, $p(\boldsymbol{x}) = p_0 + O(r^{-n})$, and (by the same proof) $\nabla\vec{\boldsymbol{w}}(\boldsymbol{x}) = O(r^{-n})$, as $r\to\infty$. Formal integration by parts, applied to the solution $\vec{\boldsymbol{w}}(\boldsymbol{x}) - \vec{\boldsymbol{w}}_0$, establishes the relation

$$\vec{\mathfrak{T}} \cdot \vec{w}_0 = 0 = 2 \int_{\mathscr{I}_R} (\operatorname{def} \vec{w})^2 \, dV + c\,(1-c) \int_{\mathscr{I}_R} p^2 \, dV + \oint_{\Sigma_R} (\vec{w} - \vec{w}_0) \cdot T \vec{w} \, dS.$$

Because of the above estimates, the outer surface integral vanishes in the limit, and we find

$$2 \int_{\mathscr{E}} (\operatorname{def} \vec{w})^2 \, dV + c\,(1-c) \int_{\mathscr{E}} p^2 \, dV = 0. \tag{6.4}$$

If $c < 3$, we verify easily, using (6.4), that $\operatorname{def} \vec{w} \equiv 0$ in $\mathscr{E}$. Hence the motion is a pure rotation. But $\vec{w} = 0$ on Σ, hence $\vec{w}(\boldsymbol{x}) \equiv 0$ in $\mathscr{E}$. This result, however, is true for every non-negative c. To see this, we observe that the divergence theorem and the above asymptotic properties of $\boldsymbol{w}(\boldsymbol{x})$ imply (summation on repeated indices)

$$\oint_{\Sigma} \left(\frac{\partial w_j}{\partial x_j} n_i - \frac{\partial w_j}{\partial x_i} n_j \right) dS = 0,$$

$$\oint_{\Sigma} (w_i - w_{0\,i}) \left(\frac{\partial w_j}{\partial x_j} n_i - \frac{\partial w_j}{\partial x_i} n_j \right) dS = \int_{\mathscr{E}} \left(\frac{\partial w_i}{\partial x_i} \frac{\partial w_j}{\partial x_j} - \frac{\partial w_i}{\partial x_j} \frac{\partial w_j}{\partial x_i} \right) dV = 0$$

since $w_i = 0$ on Σ. Adding this relation to (6.4), we find, using (1.1 b),

$$\int_{\mathscr{E}} (\nabla \vec{w})^2 \, dV + c \int_{\mathscr{E}} p^2 \, dV = 0,$$

and we conclude again $\vec{w} \equiv 0$ in $\mathscr{E}$.

If $n \geqq 3$, the relation (4.3) can be obtained by the method used to prove (6.4), the outer surface integral vanishing in the limit because of the asymptotic properties of the solution. For $n = 2$ the volume integrals which occur are necessarily infinite when $\vec{\mathfrak{T}} \neq 0$, as can be shown by using Theorem 1 and the explicit knowledge of $\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})$.

Proof of Theorem 3. By Theorem 1, it is sufficient to observe that if $\vec{\mathfrak{T}} \neq 0$, then

$$\boldsymbol{\chi} \cdot \vec{\mathfrak{T}} = \begin{cases} O(\log r), & n = 2, \\ O(r^{2-n}), & n > 2. \end{cases}$$

This property is easily checked, using the explicit knowledge of $\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})$.

Proof of Theorems 4 and 5. These theorems are proved by the methods used for Theorems 1 and 2, the differing estimates arising because of the different properties of the fundamental solution tensor in the case $\vec{\boldsymbol{b}} \neq 0$. We omit details.

Proof of Theorem 6. As in the proof of Theorem 2, we use the identity

$$\int_{\mathscr{I}_R} (\nabla \vec{w})^2 \, dV + c \int_{\mathscr{I}_R} p^2 \, dV = \oint_{\Sigma_R} \left[\vec{w} \cdot \frac{\partial \vec{w}}{\partial n} - p\,(\vec{w} \cdot \vec{n}) \right] dS.$$

According to our assumptions, the estimates of Theorem 1 or of Theorem 4 must hold with $\vec{w}_0 = 0$. It follows that the outer surface integral vanishes in the limit. Hence $\nabla \vec{w} \equiv 0$ in $\mathscr{E}$, from which $\vec{w} \equiv \text{const.}$ But $\vec{w} = 0$ on Σ, therefore $\vec{w}(\boldsymbol{x}) \equiv 0$ in $\mathscr{E}$, *q.e.d.*

Proof of Theorem 7. We first observe that by Theorem 1 or Theorem 4 ($\vec{\boldsymbol{w}}=0$ on Σ_n)

$$\vec{\boldsymbol{w}}(\boldsymbol{x}) = \vec{\boldsymbol{w}}_0 + \boldsymbol{\chi}(\boldsymbol{x}) \cdot \vec{\mathfrak{T}} + \vec{\boldsymbol{w}}_1(\boldsymbol{x})$$

where $\vec{\boldsymbol{w}}_1(\boldsymbol{x})$ has finite square integral over $\mathscr{E}$ except in the case $n=2$, $\vec{\boldsymbol{b}}=0$. (The proof of this fact requires estimates on $\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{y})$ which are slightly more delicate than those indicated in §3, but is not difficult.) Putting the exceptional case aside for the moment, we see that $\int\limits_{\mathscr{E}} |\vec{\boldsymbol{w}} - \vec{\boldsymbol{w}}_0|^2\, dV$ is finite or infinite according as $\int\limits_{\mathscr{E}} |\boldsymbol{\chi} \cdot \vec{\mathfrak{T}}|^2\, dV$ is finite or infinite. Denoting the components of $\vec{\mathfrak{T}}$ by $\{f_i\}$, the integrand becomes a quadratic form, $|\boldsymbol{\chi} \cdot \vec{\mathfrak{T}}|^2 = \chi_{ij}\chi_{ik} f_j f_k$. We adopt spherical coordinates and observe that because of the symmetry properties of χ_{ij}, the integral over a sphere centered at the origin of any term for which $j \neq k$ must vanish. Thus, we need only consider an integral of a sum of squares. Each of these terms has, however, an infinite volume integral unless $\vec{\mathfrak{T}}=0$, as is easily verified from the explicit expression for $\boldsymbol{\chi}$. If $\vec{\mathfrak{T}}=0$ on the other hand, we conclude from Theorem 2 or Theorem 5 that $\vec{\boldsymbol{w}}(\boldsymbol{x}) \equiv 0$ in $\mathscr{E}$.

It remains to discuss the exceptional case $n=2$, $\vec{\boldsymbol{b}}=0$. In this case we find that if $\vec{\mathfrak{T}} \neq 0$, the ratio of the integrals, $\int\limits_{\mathscr{S}_R} |\boldsymbol{\chi} \cdot \vec{\mathfrak{T}}|^2 dV$ and $\int\limits_{\mathscr{S}_R} |\vec{\boldsymbol{w}}_1|^2 dV$, tends to infinity with R. Thus we conclude again that $\int\limits_{\mathscr{E}} |\vec{\boldsymbol{w}}|^2 dV = \infty$ unless $\vec{\mathfrak{T}}=0$, but again $\vec{\mathfrak{T}}=0$ implies by Theorem 2 that $\vec{\boldsymbol{w}}(\boldsymbol{x}) \equiv 0$ in $\mathscr{E}$, *q.e.d.*

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On the Quasi-Static Theory of Viscoelasticity

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1. Introduction

In general, viscoelastic deformations are time dependent. However, if the inertial forces resulting from the deformation are "small", it is customary [*1*–*3*] to neglect them in the equations of motion. The resulting differential equations are those of the "quasi-static" theory which may supply an approximation to the dynamic theory. For many viscoelastic materials, neglecting the inertia terms removes the highest order derivative with respect to time from the differential equations. Therefore, one or more of the initial conditions of the dynamic theory may not be satisfied by solutions of the quasi-static differential equations. Hence there is a "boundary layer" effect in time or, more appropriately, an "initial layer". The sense of approximation of the quasi-static theory and its relation to the dynamic theory have not, to the author's knowledge, been previously studied.

In this paper we investigate these questions for a specific problem. It is the settling of a layer of a Kelvin material under a gravitational force. This is treated as a problem in the three-dimensional linear theory of viscoelasticity. We show, using a boundary layer expansion method [*4*]*, the manner in which the quasi-static solution approximates the dynamic solution for this problem. In addition, we obtain a sequence of approximating problems whose solutions give higher order or "dynamic" corrections to the quasi-static solution. Since the dynamic problem is easily solved, we can evaluate the accuracy of the approximations given by our expansion procedure.

2. Formulation

Consider an infinite layer of height $y=H$ of a Voigt or Kelvin material resting on the x, z plane and bounded by smooth vertical walls at $x=\pm L$. At time $t=0$ we assume that gravity is "switched on". We investigate the settling of this layer within the scope of the three-dimensional linear theory of dynamic viscoelasticity [*6*].

If $\sigma_{ij}(x_k, t)$, $i, j, k=1, 2, 3$**, are the components of stress and $u_i(x_k, t)$ are the components of displacement, then the equations of motion are

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + F_i = \varrho \frac{\partial^2 u_i}{\partial t^2}, \qquad i=1, 2, 3. \tag{1a}$$

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* See also a related study by WHITHAM [*5*] concerning problems in wave propagation.

** Here, $x_1=x$, $x_2=y$, $x_3=z$.

Here ϱ is the density of the material, and $F_i(x_k, t)$ are the components of the body force given by

$$F_1 = F_3 \equiv 0, \quad F_2 = -\varrho g, \tag{1b}$$

where g is the gravitational constant.

Stress-strain relations describing a Kelvin material are [6]

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu e_{ij} + \lambda' \frac{\partial e}{\partial t} \delta_{ij} + 2\mu' \frac{\partial e_{ij}}{\partial t}, \tag{2}$$

where δ_{ij} is Kronecker's delta and $e_{ij}(x_k, t)$ are the components of strain. They are given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3a}$$

and

$$e = \frac{\partial u_i}{\partial x_i}. \tag{3b}$$

The material constants λ and μ are the Lamé elastic constants, and λ' and μ' are the corresponding viscous constants. It is frequently assumed that the Kelvin material is perfectly elastic with respect to dilatational motions. Then there are only three independent material constants. This assumption is not made here since the more general form (2) produces no additional complications.

To complete the formulation appropriate boundary and initial conditions are required. We employ the following conditions: The surface $y = H$ is force free, and therefore

$$\sigma_{22}(x, H, z, t) = \sigma_{12}(x, H, z, t) = \sigma_{32}(x, H, z, t) \equiv 0. \tag{4a}$$

The walls at $x = \pm L$ are smooth and rigid so that

$$\sigma_{12}(\pm L, y, z, t) = \sigma_{13}(\pm L, y, z, t) = u_1(\pm L, y, z, t) \equiv 0. \tag{4b}$$

The material adheres to the x, z plane; hence

$$u_i(x, 0, z, t) = 0, \qquad i = 1, 2, 3. \tag{4c}$$

Appropriate initial conditions are

$$u_i(x, y, z, 0) = \frac{\partial u_i(x, y, z, 0)}{\partial t} \equiv 0, \qquad i = 1, 2, 3. \tag{5}$$

This implies that the body is at rest at $t = 0$[*]. The complete formulation of the dynamic viscoelasticity problem consists in equations (1—5).

3. The Dynamic Problem

We first introduce the dimensionless variables

$$Y = \frac{y}{H}, \quad 0 \leqq Y \leqq 1, \quad T = \left(\frac{\lambda + 2\mu}{\lambda' + 2\mu'}\right) t \tag{6a}$$

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[*] From (2), (3) and (5) it follows that the body is also stress free at $t = 0$.

and the dimensionless material parameters

$$\varepsilon^2 = \frac{\varrho H^2(\lambda+2\mu)}{(\lambda'+2\mu')^2} \geqq 0, \qquad \alpha = \frac{2\mu}{\lambda'+2\mu'}\left(\frac{\lambda'}{\lambda} - \frac{\mu'}{\mu}\right). \tag{6b}$$

Then we seek a solution of (1—5) of the form

$$u_1(x, y, z, t) = u_3(x, y, z, t) = 0, \qquad u_2(x, y, z, t) = \left(\frac{\varrho g H^2}{\lambda+2\mu}\right) v(Y, T; \varepsilon). \tag{7}$$

We find that this is a solution if $v(Y, T; \varepsilon)$ is a solution of

$$\frac{\partial^2 v}{\partial Y^2} + \frac{\partial^3 v}{\partial Y^2 \partial T} = \varepsilon^2 \frac{\partial^2 v}{\partial T^2} + 1, \tag{8a}$$

$$v(0, T; \varepsilon) = 0, \qquad \frac{\partial v(1, T; \varepsilon)}{\partial Y} + \frac{\partial^2 v(1, T; \varepsilon)}{\partial Y \partial T} = 0, \tag{8b}$$

$$v(Y, 0; \varepsilon) = \frac{\partial v(Y, 0; \varepsilon)}{\partial T} = 0. \tag{8c}$$

We further define dimensionless stress components as

$$\sigma_x(Y, T; \varepsilon) \equiv \left(\frac{\lambda+2\mu}{\lambda \varrho g H}\right)\sigma_{11}(x, y, z, t), \qquad \sigma_z(Y, T; \varepsilon) = \left(\frac{\lambda+2\mu}{\lambda \varrho g H}\right)\sigma_{33}(x, y, z, t),$$
$$\sigma_y(Y, T; \varepsilon) = (\varrho g H)^{-1}\sigma_{22}(x, y, z, t).$$

Then (2), (6) and (7) yield

$$\sigma_x = \sigma_z = \frac{\partial v}{\partial Y} + (1+\alpha)\frac{\partial^2 v}{\partial Y \partial T}, \qquad \sigma_y = \frac{\partial v}{\partial Y} + \frac{\partial^2 v}{\partial Y \partial T}. \tag{9}$$

All other stress components are identically zero.

A formal solution of (8) is

$$v(Y, T; \varepsilon) = v^e(T) + \sum_{n=1}^{\infty} A_n f_n(T) \sin \beta_n Y. \tag{10a}$$

Here $\beta_n = \frac{(2n-1)\pi}{2}$ and

$$v^e(Y) = \tfrac{1}{2}(Y^2 - 2Y) \tag{10b}$$

is the solution of the corresponding *static elastic problem*. If ε is restricted to the range $0 < \varepsilon < \pi/4$, then

$$f_n(T) = \frac{b_n \exp[a_n T] - a_n \exp[b_n T]}{b_n - a_n}, \tag{10c}$$

where

$$\begin{aligned} a_n &= -\frac{1}{8}\left[\frac{(2n-1)\pi}{\varepsilon}\right]^2 \left\{1 + \left[1 - \left(\frac{4\varepsilon}{\pi(2n-1)}\right)^2\right]^{\frac{1}{2}}\right\}, \\ b_n &= -\frac{1}{8}\left[\frac{(2n-1)\pi}{\varepsilon}\right]^2 \left\{1 - \left[1 - \left(\frac{4\varepsilon}{\pi(2n-1)}\right)^2\right]^{\frac{1}{2}}\right\}, \end{aligned} \tag{10d}$$

and

$$A_n = \frac{16}{\pi^3(2n-1)^3}$$

are the Fourier sine coefficients of $-v^e(Y)$. We observe that for ε in this range a_n and b_n are negative for all n. Thus as T increases the solution (10) decays to $v^e(Y)$. If $\varepsilon > \pi/4$, then some $f_n(T)$ have an oscillating decay in T. From (9) and (10) the dynamic viscoelastic stresses are

$$\begin{aligned} &\sigma_y(Y,T;\varepsilon)=\sigma+\sum_{n=1}^{\infty}A_n\beta_n\left[f_n(T)+\frac{df_n(T)}{dT}\right]\cos\beta_n Y,\\ &\sigma_x(Y,T;\varepsilon)=\sigma_z(Y,T;\varepsilon)=\sigma+\sum_{n=1}^{\infty}A_n\beta_n\left[f_n(T)+(1+\alpha)\,\frac{df_n(T)}{dT}\right]\cos\beta_n Y, \end{aligned} \tag{11a}$$

where

$$\sigma_y^e(Y)=\sigma_x^e(Y)=\sigma(Y)=Y-1 \tag{11b}$$

are the dimensionless static elastic stresses.

It can be shown that the infinite series in (10) and (11) converge uniformly and absolutely for $0\leqq Y\leqq 1$ and $T\geqq 0$ and that term-by-term differentiation of (10a) is permissible for the orders indicated in (8a) for all $T\geqq\delta$ where $\delta>0$. Thus the formal solution (10) is a solution of (8). Uniqueness of the solution of (8) can be shown by an "energy" argument*. In Section 6 the solution given by (10), which we call the dynamic solution, is compared with the solutions determined by the quasi-static and higher order theories.

4. Derivation of the Quasi-Static Problems

The differential equation of the quasi-static theory is obtained from the dynamic theory (8a) by setting $\varepsilon=0$. The resulting equation is of first order in time, and hence its solution cannot satisfy both initial conditions (8c). Thus there is a "boundary layer", or, more properly, an "initial layer" behavior in the variable T near $T=0$. To investigate systematically this behavior and to determine the appropriate initial condition for the quasi-static equation we assume that $v(Y,T;\varepsilon)$ can be represented asymptotically by a power series in $k=\varepsilon^2$:

$$v(Y,T;\varepsilon)\sim\sum_{n=0}^{\infty}v^n(Y,T)\,k^n. \tag{12}$$

Here the $v^n(Y,T)$ are called the quasi-static displacement coefficients, and we define $v^n=0$ if $n<0$. Similar expansions are also assumed for each stress component in (9). Substituting these expressions into (8a, b) and (9) and equating coefficients of like powers of k yields the sequence of equations:

$$\frac{\partial^2 v^n}{\partial Y^2}+\frac{\partial^3 v^n}{\partial Y^2\,\partial T}=\delta_{0n}+\frac{\partial^2 v^{n-1}}{\partial T^2}, \tag{13a}$$

$$v^n(0,T)=\frac{\partial v^n(1,T)}{\partial Y}+\frac{\partial^2 v^n(1,T)}{\partial Y\,\partial T}=0, \tag{13b}$$

$$\begin{aligned} &\sigma_x^n(Y,T)=\sigma_z^n(Y,T)=\frac{\partial v^n}{\partial Y}+(1+\alpha)\,\frac{\partial^2 v^n}{\partial Y\,\partial T},\\ &\sigma_y^n(Y,T)=\frac{\partial v^n}{\partial Y}+\frac{\partial^2 v^n}{\partial Y\,\partial T},\qquad n=0,1,\ldots. \end{aligned} \tag{13c}$$

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* The author is indebted to Professor K. O. FRIEDRICHS for bringing this to his attention.

Each of the differential equations in (13a) is of first order in T and therefore requires only one initial condition. Thus the expansion (12) cannot, in general, satisfy both initial conditions (8c). If the series (12) represents the solution of the dynamic problem, it can do so only in a T-interval away from $T=0$. The expansion does not hold uniformly up to $T=0$ and is not valid there. The solution of the dynamic theory deviates from (12) as $T\to 0$ (see also the results presented in Section 6). We call the region of sharp deviation the "initial layer". In the limit as $\varepsilon\to 0$ the initial layer vanishes (see Section 6). To obtain an expansion of the solution near $T=0$ we introduce a new independent variable, τ, by "stretching" T [*4*]. We do this so that as $\varepsilon\to 0$ the initial layer in the τ variable does not vanish. Then the resulting expansions in τ may uniformly represent the solution up to $T=0$. Let

$$\tau=\frac{T}{k}=\frac{T}{\varepsilon^2}\,. \tag{14}$$

Then any fixed neighborhood of zero in the T variable is transformed by (14) into an arbitrarily large neighborhood in the τ variable provided ε is sufficiently small.

We define the initial layer quantities,

$$\begin{aligned} V(Y,\tau;\varepsilon)&=v(Y,T;\varepsilon), \qquad S(Y,\tau;\varepsilon)=\sigma_y(Y,T;\varepsilon),\\ R(Y,\tau;\varepsilon)&=\sigma_x(Y,T;\varepsilon)=\sigma_z(Y,T;\varepsilon). \end{aligned} \tag{15}$$

It is assumed that each of them can be represented by a power series in $k=\varepsilon^2$:

$$V(Y,\tau;\varepsilon)\sim\sum_{n=0}^{\infty}V^n(Y,\tau)\,k^n, \tag{16}$$

where $V^n(Y,\tau)=0$ if $n<0$. Substituting (14—16) and the corresponding expansions for S and R into (8) and (9) and equating coefficients of like powers of k, we obtain the following initial layer problems:

$$\begin{aligned} &\frac{\partial^3 V^n}{\partial Y^2\,\partial\tau}-\frac{\partial^2 V^n}{\partial\tau^2}=\delta_{1n}-\frac{\partial^2 V^{n-1}}{\partial Y^2},\\ &V^n(0,\tau)=\frac{\partial^2 V^n(1,\tau)}{\partial Y\,\partial\tau}+\frac{\partial V^{n-1}(1,\tau)}{\partial Y}=0,\\ &V^n(Y,0)=\frac{\partial V^n(Y,0)}{\partial\tau}=0,\\ &S^{n-1}=\frac{\partial V^{n-1}}{\partial Y}+\frac{\partial^2 V^n}{\partial Y\,\partial\tau},\qquad R^{n-1}=\frac{\partial V^{n-1}}{\partial Y}+(1+\alpha)\frac{\partial^2 V^n}{\partial Y\,\partial\tau},\qquad n=0,1,\ldots. \end{aligned} \tag{17}$$

In addition, we require that the initial layer coefficients approach or "match" the quasi-static coefficients for large τ. More specifically we assume that the $v^n(Y,T)$ are regular functions of T near $T=0$ and expand them in a Taylor series,

$$v^n(Y,T)=\sum_{m=0}^{\infty}p_m^n(Y)\,T^m,$$

where

$$p_m^n(Y) = \begin{cases} \dfrac{1}{m!}\,\dfrac{\partial^m v^n(Y,0)}{\partial T^m}, & n \geqq 0, \\ 0, & n < 0. \end{cases} \tag{18a}$$

We find from (12) and (14) that near $T=0$

$$v(Y,T;\varepsilon) \sim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_m^n(Y)\, T^m k^n = \sum_{n=0}^{\infty} U^n(Y,\tau)\, k^n, \tag{18b}$$

where

$$U^n(Y,\tau) = \sum_{m=0}^{n} p_m^{n-m}(Y)\, \tau^m \tag{18c}$$

are the quasi-static displacement coefficients near $T=0$ as functions of Y and τ. Employing (15), (16) and (18), the matching conditions or the asymptotic forms for the V^n are obtained by identifying each V^n with the corresponding U^n as $\varepsilon \to 0$. Using (14), we write these conditions as

$$\lim_{\tau\to\infty} [V^n(Y,\tau) - U^n(Y,\tau)] = 0, \qquad n = 0, 1, \ldots. \tag{18d}$$

For example, by first setting $n=0$ and then $n=1$ in (18) we find that

$$v^0(Y,0) = \lim_{\tau\to\infty} V^0(Y,\tau), \qquad v^1(Y,0) = \lim_{\tau\to\infty}\left[V^1(Y,\tau) - \frac{\partial v^0(Y,0)}{\partial T}\,\tau\right]. \tag{19}$$

Equations (19) yield the appropriate initial conditions for $v^0(Y,T)$ and $v^1(Y,T)$ in terms of the asymptotic form of the solutions of the initial layer problems, (17), of order zero and one. Appropriate initial conditions for $v^n(Y,T)$ with $n > 1$ are obtained in a similar manner from (18).

5. The Asymptotic Form of $V(Y,\tau)$

To obtain, from (18), the proper initial values for the quasi-static problems (13) we require the asymptotic forms in τ of V^n. These are obtained from the solutions of the initial layer problems (17). The zero$^{\text{th}}$ order initial layer problem is given by (17) with* $n=0$ and has the solution

$$V^0(Y,\tau) = 0. \tag{20}$$

Therefore, from the first of (19) the appropriate initial condition for $v^0(Y,T)$ is

$$v^0(Y,0) = 0. \tag{21}$$

The first-order initial layer problem is given by (20) and (17) with $n=1$:

$$\begin{gathered} \frac{\partial^3 V^1}{\partial Y^2\,\partial\tau} - \frac{\partial^2 V^1}{\partial\tau^2} = 1, \\ V^1(0,\tau) = \frac{\partial^2 V^1(1,\tau)}{\partial Y\,\partial\tau} = 0, \\ V^1(Y,0) = \frac{\partial V^1(Y,0)}{\partial\tau} = 0. \end{gathered} \tag{22}$$

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* We recall that $V^n = R^n = S^n \equiv 0$ if $n < 0$.

By employing Laplace transform techniques, we find that an integral representation of the solution of (22) is

$$V^1(Y,\tau) - \int_{c \quad i\infty}^{c \quad i\infty} G(Y,p)\, e^{p\tau}\, dp, \tag{23 a}$$

where

$$G(Y,p) = \frac{1}{p^3}\left[\frac{\cosh(1-Y)\,p^{\frac{1}{2}}}{\cosh p^{\frac{1}{2}}} - 1\right] \tag{23 b}$$

and c is chosen so that the path of integration is to the right of the singularities of $G(Y,p)$. The integral in (23) can be evaluated in terms of theta functions [7], yielding an approximation to the dynamic solution near $T=0$. However, to obtain from the second of (19) the correct initial condition for $v^1(Y,T)$ we require $V^1(Y,\tau)$ for large τ. This is easily evaluated from (23) by standard methods [8]. Substituting the result of this calculation into the second of (19), we obtain the initial condition for $v^1(Y,T)$ as

$$v^1(Y,0) = \frac{1}{4!}\left[Y^4 - 4Y^3 + 8Y\right], \tag{24}$$

and the auxiliary condition,

$$\frac{\partial v^0(Y,0)}{\partial T} = \frac{1}{2}(Y^2 - 2Y).$$

The latter condition is automatically satisfied; see (25 a) below.

The initial conditions (21) and (24) complete the formulation of the quasi-static problems of the two lowest orders. The "zero$^{\text{th}}$ order quasi-static problem", *i.e.* the quasi-static theory consisting of (21) and (13) with $n=0$, has the solution

$$v^0(Y,T) = \tfrac{1}{2}(Y^2 - 2Y)(1 - e^{-T}), \tag{25 a}$$

$$\sigma_y^0(Y,T) = (Y-1), \qquad \sigma_x^0(Y,T) = \sigma_z^0(Y,T) = (Y-1)(1 + \alpha e^{-T}). \tag{25 b}$$

The "first-order quasi-static problem" consisting in (24) and (13) with $n=1$ has the solution

$$v^1(Y,T) = -f(Y)(1-T)\,e^{-T}, \tag{26 a}$$

$$\sigma_y^1(Y,T) = \frac{df(Y)}{dY}\, e^{-T}, \qquad \sigma_x^1(Y,T) = \sigma_z^1(Y,T) = \sigma_y^1(Y,T)\left[1 - \alpha(T-2)\right], \tag{26 b}$$

where

$$f(Y) = -\frac{Y}{6}\left(\frac{Y^3}{4} - Y^2 + 2\right). \tag{26 c}$$

6. Discussion of Results

The first approximation, valid away from $T=0$, to the solution of the dynamic problem, (10) and (11), is given by (25). We observe from (9), (10 b) and (11 b) that for the *static elastic* problem

$$\sigma_{11}^e = \frac{\lambda}{\lambda + 2\mu}\,\sigma_{22}^e < \sigma_{22}^e.$$

The viscoelastic stress σ_y^0, equation (25b), "jumps" at $T=0$ to the static elastic value. Thus, within the first approximation σ_y shows no viscoelastic effect. The behavior of σ_x^0 and σ_z^0 depend strongly upon α; see Fig. 1. For $\alpha=0$★, $\sigma_x^0=\sigma_y^0$, and there are no viscoelastic effects. However, for $\alpha>0$, σ_x^0 "jumps" at $T=0$ to $\sigma_x^0=(Y-1)(1+\alpha)$ and then approaches its static elastic value from above; for $\alpha<0$ the approach is from below. Therefore, according to the quasi-static theory for $\alpha>0$, $\sigma_x^0>\sigma_x^e$ and σ_x^0 is the maximum stress. This is in contrast to the static elastic case. In Fig. 1, $\sigma_x^0(0, T)$ and the corresponding dynamic stress are given as functions of T for $\varepsilon=\pi/5$ and varying α. From this figure we observe that for $\alpha>0$ the accuracy of approximation of $\sigma_x^0(0, T)$ decreases as T decreases. When $T<1.5$, σ_x^0 is, in general, a poor approximation. For $\varepsilon=\pi/10$ (not shown) the quasi-static solution is an accurate approximation for $T>.25$ and $-1\leq\alpha\leq 1$.

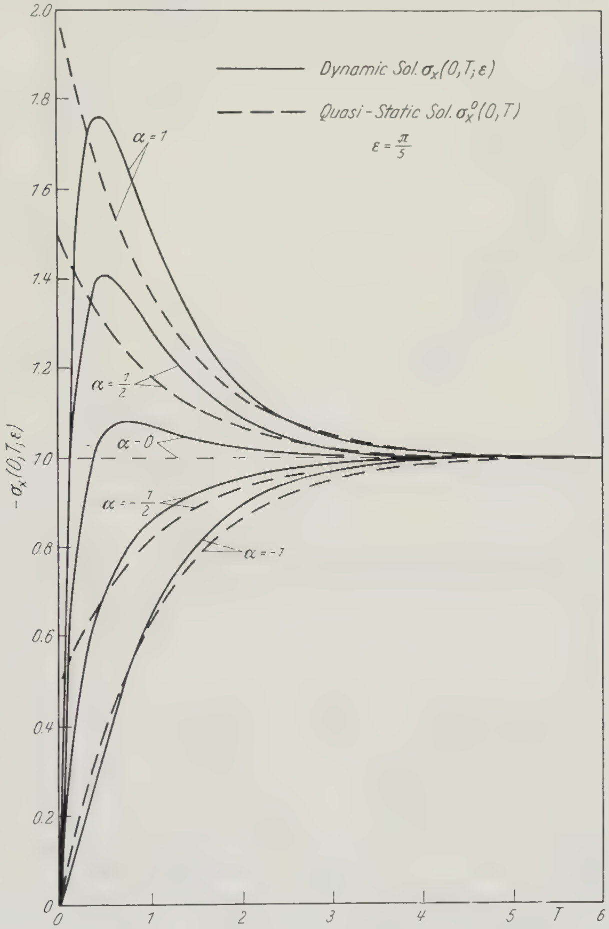

Fig. 1. The T-variation for fixed ε and varying α of σ_x obtained from the dynamic and quasi-static solutions

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★ I.e. $\lambda'=\frac{\mu'}{\mu}\lambda$.

For $\varepsilon = \pi/100$ (not shown) the approximation is accurate for $T > .01$ and the same range of α. In Figs. 2 and 3 the quasi-static and dynamic solutions are compared for fixed α and varying ε. The vanishing of the initial layer as $\varepsilon \to 0$ is illustrated in these figures.

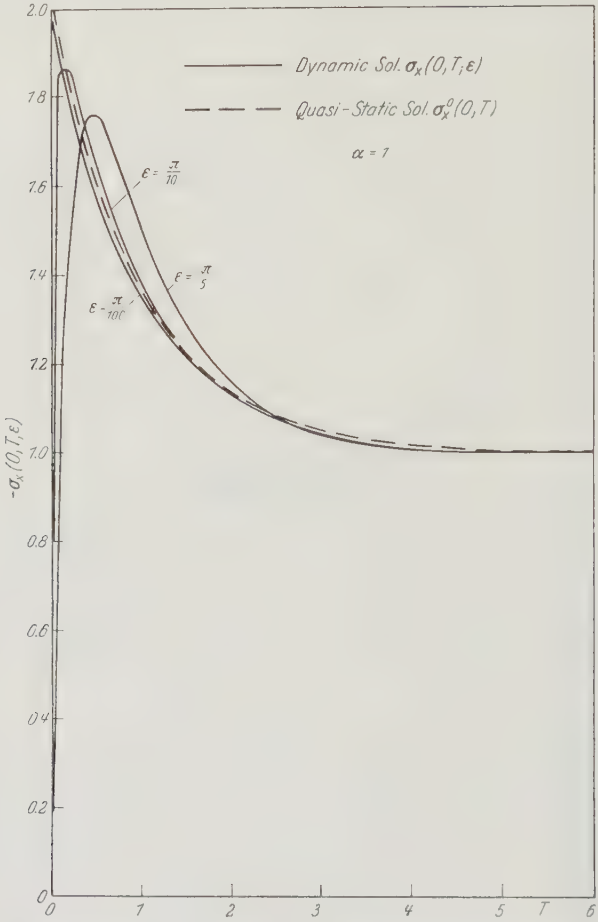

Fig. 2. The T-variation for fixed α and varying ε of σ_x obtained from the dynamic and quasi-static solutions

To obtain more accurate approximations for "larger" ε and T away from zero, the solutions of the zero^th^ and first order quasi-static problems, (24, 25), are combined according to (12) to form:

$$\sigma_y^{(1)} = \sigma_y^0 + \varepsilon^2 \sigma_y^1, \qquad \sigma_x^{(1)} = \sigma_z^{(1)} = \sigma_x^0 + \varepsilon^2 \sigma_x^1.$$

In Fig. 3, the improved accuracy of $\sigma_y^{(1)}$ as compared with σ_y^0 is apparent.

More accurate approximations to the dynamic solution for T near zero may be obtained from the solutions of the initial layer problems, (17).

In conclusion we note that our approximation procedure can also be obtained directly from the dynamic solution. Our method is advantageous for those problems whose dynamic solution cannot be found explicitly. Increasingly accurate approximations may then be obtained* by solving a sequence of comparatively simple problems.

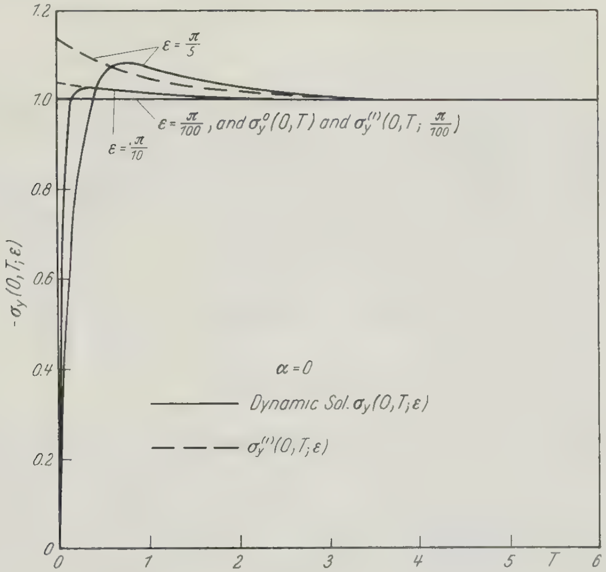

Fig. 3. The T-variation of σ_y obtained from the dynamic, the quasi-static and the "improved" quasi-static solution

The work reported in this paper was performed under Contract No. AT(30-1)-1480 with the United States Atomic Energy Commission. The author is indebted to H. B. Keller for his many helpful suggestions and stimulating discussions.

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* Provided ε is sufficiently small.

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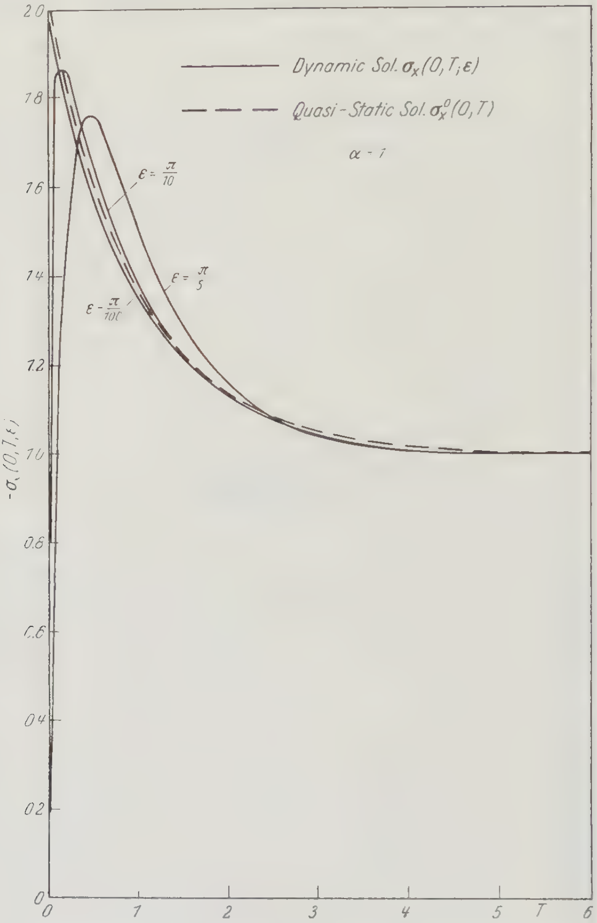

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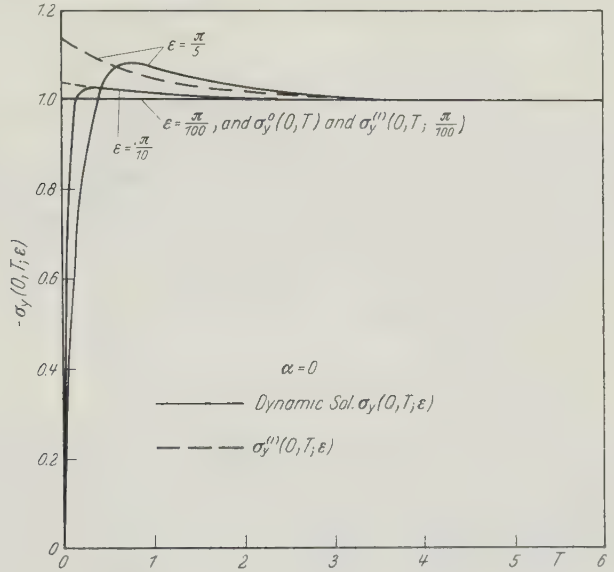

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Eine Verschärfung der asymptotischen Gesetze elektromagnetischer Hohlraumschwingungen

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Vorgelegt von C. Müller

Einleitung

Es sei G ein reguläres[1] Gebiet des dreidimensionalen euklidischen Raumes, das mit homogener und isotroper Materie erfüllt ist und das von einer im physikalischen Sinne ideal leitenden, dreimal stetig differenzierbaren Randfläche F begrenzt wird. Die Materialeigenschaften seien durch die Dielektrizitätskonstante ε und die Permeabilität μ festgelegt. Die Konstanten ε und μ seien positiv reell.

Elektromagnetische Eigenschwingungen $\mathfrak{E}$, $\mathfrak{H}$ zur Eigenfrequenz ω genügen in G den Gleichungen

$$\nabla\times\mathfrak{H}+i\omega\,\varepsilon\,\mathfrak{E}=\mathfrak{o},\qquad \nabla\times\mathfrak{E}-i\omega\,\mu\,\mathfrak{H}=\mathfrak{o}$$

und der Randbedingung $\mathfrak{n}\times\mathfrak{E}=\mathfrak{o}$ auf F. Dabei ist $\mathfrak{n}$ der ins Äußere von G weisende Normalenvektor auf F. Mathematisch stellt dies ein Eigenwertproblem für ein System von sechs partiellen Differentialgleichungen für die insgesamt sechs Komponentenfunktionen der Vektoren $\mathfrak{E}$ und $\mathfrak{H}$ dar mit einer homogenen Randbedingung für die Komponenten von $\mathfrak{E}$.

In einer früheren Arbeit gemeinsam mit C. Müller [9] wurde gezeigt, daß dieses Eigenwertproblem unendlich viele Eigenfrequenzen $\omega_1, \omega_2, \omega_3, \ldots$ besitzt, die alle reell sind, sich im Endlichen nicht häufen und jeweils von endlicher Vielfachheit sind. Der Fall $\omega=0$ ist hierbei auszuschließen. Die zugehörigen Eigenschwingungen $\mathfrak{E}_n$ und $\mathfrak{H}_n$ sind in G stetig differenzierbar, im abgeschlossenen Bereich $\overline{G}$ stetig und erfüllen die Randbedingung $\mathfrak{n}\times\mathfrak{E}_n=\mathfrak{o}$ auf F. Jeder Eigenfrequenz ω_n entspricht eine Eigenfrequenz $-\omega_n$ gleicher Vielfachheit, wobei die Eigenschwingungen zur Eigenfrequenz $-\omega_n$ konjugiert komplex zu denen von ω_n sind. Wir betrachten daher nur die positiven Eigenfrequenzen, die wir uns monoton wachsend angeordnet denken wollen $(0<\omega_1\leqq\omega_2\leqq\cdots)$.

In der erwähnten Arbeit wurden für die so normierten Eigenschwingungen und Eigenfrequenzen unter anderem folgende asymptotische Aussagen bewiesen

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[1] „regulär" im Sinne von O. D. Kellog [6], S. 113.

($\mathfrak{x}\in G,\ x\to\infty$):
$$\sum_{0<\omega_n\leqq x}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2}=\mu(\varepsilon\mu)^{\frac{3}{2}}\,\frac{x}{\pi^2}+o(x)$$
$$\sum_{0<\omega_n\leqq x}|\mathfrak{E}_n(\mathfrak{x})|^2=\mu(\varepsilon\mu)^{\frac{3}{2}}\,\frac{x^3}{3\pi^2}+o(x^3)$$
$$\sum_{0<\omega_n\leqq x}1=(\varepsilon\mu)^{\frac{3}{2}}\frac{V\cdot x^3}{3\pi^2}+o(x^3).$$

Hierbei ist V das Volumen des betrachteten Gebietes G, und $\sum\limits_{0<\omega_n\leqq x}1$ bedeutet die Anzahl der positiven Eigenfrequenzen ω_n, die kleiner oder gleich x sind.

Wir werden in dieser Arbeit zeigen, daß die Fehlerglieder in diesen asymptotischen Aussagen verbessert werden können, und zwar beweisen wir ($\mathfrak{x}\in G$, $\mathfrak{y}\in G$)
$$\sum_{0<\omega_n\leqq x}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2}=\mu(\varepsilon\mu)^{\frac{3}{2}}\frac{x}{\pi^2}+O(1)$$
$$\sum_{0<\omega_n\leqq x}|\mathfrak{E}_n(\mathfrak{x})|^2=\mu(\varepsilon\mu)^{\frac{3}{2}}\,\frac{x^3}{3\pi^2}+O(x^2)$$
$$\sum_{0<\omega_n\leqq x}\frac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}_n^{\sim}(\mathfrak{y})}{\omega_n^2}=O(1)\quad\text{für}\quad\mathfrak{x}\neq\mathfrak{y}$$
und
$$\sum_{0<\omega_n\leqq x}1=(\varepsilon\mu)^{\frac{3}{2}}\frac{V\cdot x^3}{3\pi^2}+O(x^2\log x).$$

In der vorletzten Gleichung ist dabei der als Spalte geschriebene Vektor $\mathfrak{E}_n(\mathfrak{x})$ mit dem transponierten Vektor $\overline{\mathfrak{E}}_n^{\sim}(\mathfrak{y})$ nach den Regeln des Matrizenkalküls zu multiplizieren[1]. In der entstehenden Matrizensumme bleibt dann jede Komponente beschränkt für $x\to\infty$.

Asymptotische Aussagen über die Eigenwerte der Helmholtzschen Schwingungsgleichung sind zuerst von H. Weyl bewiesen worden [*10*], [*11*], [*12*]. Im Jahre 1934 konnte T. Carleman [*3*], [*4*] darüber hinaus auch für die Eigenfunktionen asymptotische Beziehungen herleiten.

Sind u_n die zur Randbedingung $u=0$ auf F gehörigen orthonormierten Eigenfunktionen der (dreidimensionalen) Schwingungsgleichung und $0<\lambda_1\leqq\lambda_2\leqq\cdots$ die zugehörigen Eigenwerte, so gilt nach T. Carleman
$$\sum_{\lambda_n\leqq x}\frac{u_n^2}{\lambda_n}=\frac{\sqrt{x}}{2\pi^2}+o(\sqrt{x})$$
$$\sum_{\lambda_n\leqq x}u_n^2=\frac{x^{\frac{3}{2}}}{6\pi^2}+o(x^{\frac{3}{2}})$$
$$\sum_{\lambda_n\leqq x}1=V\cdot\frac{x^{\frac{3}{2}}}{6\pi^2}+o(x^{\frac{3}{2}}).$$

Die letztgenannte Abschätzung der Anzahl der Eigenwerte ist die ursprüngliche Weylsche Formel, die später von R. Courant mit Methoden der Variationsrechnung verbessert wurde. Courant konnte zeigen, daß das Fehlerglied in dieser Formel nur von der Ordnung $O(x\cdot\log x)$ ist.

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[1] Transponieren deuten wir durch $\sim$ an, den Übergang zum konjugiert komplexen durch $\bar{\ }$.

V. G. AVAKUMOVIĆ [*1*], [*2*] verschärfte 1952 auch die asymptotischen Beziehungen über die Eigenfunktionen und bewies:

$$\sum_{\lambda_n \leq x} \frac{u_n^2(\mathfrak{x})}{\lambda_n} = \frac{\sqrt{x}}{2\pi^2} + O(1)$$

$$\sum_{\lambda_n \leqq x} \frac{u_n(\mathfrak{x})\, u_n(\mathfrak{y})}{\lambda_n} = O(1) \qquad (\mathfrak{x} \neq \mathfrak{y})$$

$$\sum_{\lambda_n \leqq x} u_n^2(\mathfrak{x}) = \frac{x^{\frac{3}{2}}}{6\pi^2} + O(x).$$

Entsprechende Relationen sind von AVAKUMOVIĆ auch für die Eigenfunktionen der Schwingungsgleichung auf geschlossenen Riemannschen Mannigfaltigkeiten abgeleitet worden. Im Fall geschlossener Riemannscher Mannigfaltigkeiten gelang es ihm auch, durch Integration der letzten Gleichung für die Anzahl der Eigenwerte die Formel

$$\sum_{\lambda_n \leqq x} 1 = \frac{V \cdot x^{\frac{3}{2}}}{6\pi^2} + O(x)$$

zu erhalten. Im Falle berandeter Mannigfaltigkeiten ist jedoch ein besseres als das Courantsche Ergebnis bis jetzt nicht bekannt.

Wir zeigen hier, daß entsprechende verschärfte Aussagen für die elektromagnetischen Eigenschwingungen gelten. Der Beweisgedanke, der in ähnlicher Weise von T. CARLEMAN und V. G. AVAKUMOVIĆ angewandt worden ist, geht von Bilinearentwicklungen aus, die die Eigenschwingungen mit den Greenschen Funktionen bzw. Greenschen Tensoren verknüpfen. Im Fall der Schwingungsgleichung kann man z.B. von Bilinearentwicklung

$$\tau^2 \sum_{n=1}^{\infty} \frac{u_n(\mathfrak{x})\, u_n(\mathfrak{y})}{\lambda_n(\lambda_n + \tau^2)} = G(\mathfrak{x}, \mathfrak{y}; 0) - G(\mathfrak{x}, \mathfrak{y}; -\tau^2)$$

ausgehen, wobei $G(\mathfrak{x}, \mathfrak{y}; -\tau^2)$ die Greensche Funktion der Schwingungsgleichung $(\Delta - \tau^2)u = 0$ mit der Randbedingung $u = 0$ auf F ist.

Setzen wir

$$\Theta(x) = \sum_{\lambda_n \leqq x} \frac{u_n(\mathfrak{x})\, u_n(\mathfrak{y})}{\lambda_n},$$

so ist

$$\tau^2 \sum_{n=1}^{\infty} \frac{u_n(\mathfrak{x})\, u_n(\mathfrak{y})}{\lambda_n(\lambda_n + \tau^2)} = \tau^2 \int_0^{\infty} \frac{d\,\Theta(x)}{x + \tau^2} = G(\mathfrak{x}, \mathfrak{y}; 0) - G(\mathfrak{x}, \mathfrak{y}; -\tau^2).$$

Wir können daher die Bilinearentwicklung als Stieltjes-Transformation der Funktion $\Theta(x)$ auffassen. Wollen wir aus dem Verhalten der Greenschen Funktion für $\tau \to \infty$ auf das Verhalten von $\Theta(x)$ für $x \to \infty$ schließen, so führt das auf einen Fragenkreis, der durch den Begriff „Taubersche Sätze" gekennzeichnet ist. Wir werden hier einen Tauberschen Satz für die Laplace-Transformationen benutzen, der von A. E. INGHAM [*5*] 1935 bewiesen worden ist.

In §1 stellen wir die wichtigsten Eigenschaften der Greenschen Tensoren für die Maxwellschen Gleichungen zusammen, die wir späterhin benötigen werden. In §2 wird das Verhalten der Bilinearentwicklung bzw. der Greenschen Tensoren für $\tau \to \infty$ untersucht und in §§3 und 4 wird der erwähnte Taubersche Satz auf die Bilinearentwicklung angewandt.

§ 1. Greensche Tensoren

In diesem Paragraphen stellen wir die wichtigsten Eigenschaften der Greenschen Tensoren für die Maxwellschen Gleichungen zusammen, die wir später benötigen. Die Ergebnisse sind in der erwähnten Arbeit [*9*] enthalten.

Es sei G ein endliches, reguläres Gebiet, das von der orientierbaren, zusammenhängenden, dreimal stetig differenzierbaren Randfläche F berandet wird. In G betrachten wir das System der Maxwellschen Gleichung

$$\nabla\times\mathfrak{H}+i\omega\,\varepsilon\,\mathfrak{E}=\mathfrak{o}\,,\qquad \nabla\times\mathfrak{E}-i\omega\,\mu\,\mathfrak{H}=\mathfrak{o} \tag{1.1}$$

mit der Randbedingung $\mathfrak{n}\times\mathfrak{E}=\mathfrak{o}$ auf F. Wir setzen ε und μ als positiv reell voraus.

Wie bereits bemerkt, gibt es unendlich viele (reelle) Eigenfrequenzen ω_n und zugehörige Eigenschwingungen $\mathfrak{E}_n$, $\mathfrak{H}_n$, wobei wir uns auf die positiven Eigenfrequenzen beschränken und diese monoton wachsend anordnen wollen. Sind ε und μ von 1 verschieden, so können wir die Gleichungen (1.1) durch die Substitutionen

$$\mathfrak{H}=\sqrt{\varepsilon}\,\hat{\mathfrak{H}}\,,\qquad \mathfrak{E}=\sqrt{\mu}\,\hat{\mathfrak{E}}\,,\qquad \omega\sqrt{\varepsilon\,\mu}=\hat{\omega} \tag{1.2}$$

zu

$$\nabla\times\hat{\mathfrak{E}}-i\hat{\omega}\,\hat{\mathfrak{H}}=\mathfrak{o}\,,\qquad \nabla\times\hat{\mathfrak{H}}+i\hat{\omega}\,\hat{\mathfrak{E}}=\mathfrak{o} \tag{1.3}$$

mit der Randbedingung $\mathfrak{n}\times\hat{\mathfrak{E}}=\mathfrak{o}$ auf F normieren. Wir setzen daher weiterhin $\varepsilon=\mu=1$ voraus.

Greensche Tensoren $\mathfrak{G}_{ik}(\mathfrak{x},\mathfrak{y};\omega)$ $(i,\,k=1,\,2)$ zu den Maxwellschen Gleichungen können durch folgende Relationen definiert werden:

1. Ihre singulären Anteile $\mathfrak{G}^*_{ik}(\mathfrak{x},\mathfrak{y};\omega)$ sind durch die Forderung bestimmt, daß für jeden konstanten Vektor $\mathfrak{a}$ und mit $\Phi(\mathfrak{x},\mathfrak{y})=\dfrac{e^{i\omega|\mathfrak{x}-\mathfrak{y}|}}{|\mathfrak{x}-\mathfrak{y}|}$

$$\begin{aligned}
&\mathfrak{G}^*_{11}(\mathfrak{x},\mathfrak{y};\omega)\,\mathfrak{a}=i\omega\,\Phi\,\mathfrak{a}+\frac{i}{\omega}\nabla_{\mathfrak{x}}(\mathfrak{a}\,\nabla_{\mathfrak{x}}\Phi)\\
&\mathfrak{G}^*_{21}(\mathfrak{x},\mathfrak{y};\omega)\,\mathfrak{a}=\nabla_{\mathfrak{x}}\Phi\times\mathfrak{a}\\
&\mathfrak{G}^*_{12}(\mathfrak{x},\mathfrak{y};\omega)\,\mathfrak{a}=-\nabla_{\mathfrak{x}}\Phi\times\mathfrak{a}\\
&\mathfrak{G}^*_{22}(\mathfrak{x},\mathfrak{y};\omega)\,\mathfrak{a}=i\omega\,\Phi\,\mathfrak{a}+\frac{i}{\omega}\nabla_{\mathfrak{x}}(\mathfrak{a}\,\nabla_{\mathfrak{x}}\Phi)
\end{aligned} \tag{1.4}$$

ist.

2. Die kompensierenden Anteile

$$\mathfrak{K}_{ik}(\mathfrak{x},\mathfrak{y};\omega)=\mathfrak{G}_{ik}(\mathfrak{x},\mathfrak{y};\omega)-\mathfrak{G}^*_{ik}(\mathfrak{x},\mathfrak{y};\omega) \tag{1.5}$$

sind in jeder Komponente als Funktion von $\mathfrak{x}$ stetig differenzierbar in G.

3. Die Felder $\mathfrak{G}_{1k}\,\mathfrak{a}$ und $\mathfrak{G}_{2k}\,\mathfrak{a}$ $(k=1,\,2)$ erfüllen als Funktionen von $\mathfrak{x}$ für $\mathfrak{x}\neq\mathfrak{y}$ die homogenen Maxwellschen Gleichungen, d.h.

$$\begin{aligned}
&\nabla_{\mathfrak{x}}\times\mathfrak{G}_{1k}\,\mathfrak{a}-i\omega\,\mathfrak{G}_{2k}\,\mathfrak{a}=\mathfrak{o}\\
&\nabla_{\mathfrak{x}}\times\mathfrak{G}_{2k}\,\mathfrak{a}+i\omega\,\mathfrak{G}_{1k}\,\mathfrak{a}=\mathfrak{o}\,.
\end{aligned} \tag{1.6}$$

4. Es gelten die Randbedingungen

$$\mathfrak{n}(\mathfrak{x}')\times\mathfrak{G}_{1k}(\mathfrak{x}',\mathfrak{y};\omega)\,\mathfrak{a}=\mathfrak{o}\qquad(\mathfrak{x}'\in F,\ \mathfrak{y}\in G)\,. \tag{1.7}$$

Ist ω keine Eigenfrequenz der Maxwellschen Gleichung — insbesondere also für rein imaginäre Werte von ω —, so sind die Greenschen Tensoren eindeutig bestimmt. Der Existenznachweis der Greenschen Tensoren führt auf ein Randwertproblem der Maxwellschen Gleichungen, das von C. MÜLLER [7], [8] bereits gelöst worden ist.

Für die kompensierenden Anteile macht man hierzu mit einem noch unbekannten Tangentialfeld $\mathfrak{v}(\mathfrak{p}, \mathfrak{y})$ an F ($\mathfrak{p} \in F$, $\mathfrak{y} \in G$) den Ansatz

$$\mathfrak{K}_{12}(\mathfrak{x}, \mathfrak{y}; \omega)\, \mathfrak{a} = -\frac{1}{4\pi} \int\limits_F \mathfrak{v}(\mathfrak{p}, \mathfrak{y}) \times \nabla_{\mathfrak{p}} \Phi(\mathfrak{p}, \mathfrak{x})\, dF_{\mathfrak{y}}. \tag{1.8}$$

Die Randbedingung Gl. (1.7) liefert dann nach den Sprungrelationen die Integralgleichung

$$\mathfrak{v}(\mathfrak{p}, \mathfrak{y}) - \frac{1}{2\pi} \int\limits_F \mathfrak{n}(\mathfrak{p}) \times \big(\mathfrak{v}(\mathfrak{q}, \mathfrak{y}) \times \nabla_{\mathfrak{q}} \Phi(\mathfrak{q}, \mathfrak{p})\big)\, dF_{\mathfrak{q}} = 2\mathfrak{n}(\mathfrak{p}) \times [\nabla_{\mathfrak{p}} \Phi(\mathfrak{p}, \mathfrak{y}) \times \mathfrak{a}] \tag{1.9}$$

symbolisch

$$\mathfrak{v} - M\mathfrak{v} = \mathfrak{w}(\mathfrak{p}, \mathfrak{y}) \qquad \big(= 2\mathfrak{n}(\mathfrak{p}) \times [\nabla_{\mathfrak{p}} \Phi \times \mathfrak{a}]\big) \tag{1.10}$$

geschrieben.

Führen wir als Norm des Flächenfeldes $\mathfrak{v}(\mathfrak{p}, \mathfrak{y})$

$$\|\mathfrak{v}\| = \operatorname*{supr}_{\mathfrak{p} \in F} |\mathfrak{v}(\mathfrak{p}, \mathfrak{y})|$$

ein, so ist die Integraltransformation M vollstetig im Banach-Raum aller stetigen Flächenfelder auf F. Ist ω keine Eigenfrequenz der Gl. (3), so ist die Integralgleichung (10) für beliebig stetige Flächenfelder eindeutig lösbar. Dieselbe Integralgleichung mit anderem inhomogenen stetigen Flächenfeld erhält man für das erzeugende Feld des kompensierenden Anteils $\mathfrak{K}_{11}\,\mathfrak{a}$.

Die mit den Lösungen dieser Integralgleichung gewonnenen Tensoren $\mathfrak{K}_{11}(\mathfrak{x}, \mathfrak{y}; \omega)$ und $\mathfrak{K}_{12}(\mathfrak{x}, \mathfrak{y}; \omega)$, sowie die daraus durch Differentiation gewonnenen Tensoren $\mathfrak{K}_{21}(\mathfrak{x}, \mathfrak{y}; \omega)$ und $\mathfrak{K}_{22}(\mathfrak{x}, \mathfrak{y}; \omega)$ erfüllen alle Forderungen in Gl. (1.4) bis (1.7).

Für die Greenschen Tensoren gelten — ebenso wie für die singulären und kompensierenden Anteile allein — folgende Symmetrierelationen

$$\mathfrak{G}_{\widetilde{11}}(\mathfrak{x}, \mathfrak{y}; \omega) = \mathfrak{G}_{11}(\mathfrak{y}, \mathfrak{x}; \omega), \tag{1.11}$$

$$\mathfrak{G}_{\widetilde{12}}(\mathfrak{x}, \mathfrak{y}; \omega) = -\mathfrak{G}_{21}(\mathfrak{y}, \mathfrak{x}; \omega), \tag{1.12}$$

$$\mathfrak{G}_{\widetilde{22}}(\mathfrak{x}, \mathfrak{y}; \omega) = \mathfrak{G}_{22}(\mathfrak{y}, \mathfrak{x}; \omega). \tag{1.13}$$

Schließlich kann man zeigen, daß die Tensoren $\mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{y}; \omega)$ und $\mathfrak{G}_{21}(\mathfrak{x}, \mathfrak{y}; \omega)$ auch noch — in einem gewissen ausgearteten Sinn — für $\omega = 0$ existieren.

Bildet man nun mit den Tensoren $\mathfrak{G}_{12}$ und $\mathfrak{G}_{21}$ folgende iterierte Tensoren

$$\mathfrak{G}_E(\mathfrak{x}, \mathfrak{y}; \omega) = -\frac{1}{(4\pi)^2} \int\limits_G \mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}(\mathfrak{z}, \mathfrak{y}; \omega)\, dV_{\mathfrak{z}}, \tag{1.14}$$

$$\mathfrak{G}_H(\mathfrak{x}, \mathfrak{y}; \omega) = -\frac{1}{(4\pi)^2} \int\limits_G \mathfrak{G}_{21}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{12}(\mathfrak{z}, \mathfrak{y}; \omega)\, dV_{\mathfrak{z}}, \tag{1.15}$$

so erfüllen die elektromagnetischen Eigenschwingungen die Integralgleichungen

$$(\omega_n^2+\tau^2)\int_G \mathfrak{G}_E(\mathfrak{x},\mathfrak{y};i\tau)\,\mathfrak{E}_n(\mathfrak{y})\,dV_{\mathfrak{y}}=\mathfrak{E}_n(\mathfrak{x})\,,\tag{1.16}$$

$$(\omega_n^2+\tau^2)\int_G \mathfrak{G}_H(\mathfrak{x},\mathfrak{y};i\tau)\,\mathfrak{H}_n(\mathfrak{y})\,dV_{\mathfrak{y}}=\mathfrak{H}_n(\mathfrak{x})\,,\tag{1.17}$$

und es gelten die Bilinearentwicklungen

$$\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};0)-\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};i\tau)=\tau^2\sum_{\omega_n>0}\frac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}_n^{\sim}(\mathfrak{y})}{\omega_n^2(\omega_n^2+\tau^2)}\,,\tag{1.18}$$

$$\mathfrak{G}_H(\mathfrak{x},\mathfrak{y};0)-\mathfrak{G}_H(\mathfrak{x},\mathfrak{y};i\tau)=\tau^2\sum_{\omega_n>0}\frac{\mathfrak{H}_n(\mathfrak{x})\,\overline{\mathfrak{H}}_n^{\sim}(\mathfrak{y})}{\omega_n^2(\omega_n^2+\tau^2)}\,.\tag{1.19}$$

§ 2. Die Bilinearentwicklung

Den Ausgangspunkt unserer Betrachtungen bildet die Bilinearentwicklung Gl. (1.18)

$$\tau^2\sum_{\omega_n>0}\frac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}_n^{\sim}(\mathfrak{y})}{\omega_n^2(\omega_n^2+\tau^2)}=\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};0)-\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};i\tau)\,.\tag{2.1}$$

Lassen wir $\mathfrak{y}\to\mathfrak{x}$ streben und bilden die Spur der Matrizen auf beiden Seiten, so ergibt sich

$$\tau^2\sum_{\omega_n>0}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2(\omega_n^2+\tau^2)}=\lim_{\mathfrak{y}\to\mathfrak{x}}\operatorname{Spur}\{\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};0)-\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};i\tau)\}\,.\tag{2.2}$$

Wir wollen das Verhalten der rechten Seite für $\tau\to\infty$ untersuchen. Dazu zerlegen wir die Tensoren nach Gl. (1.4) und (1.5) in ihre singulären und kompensierenden Bestandteile und erhalten drei Summanden, die wir mit R_1, R_2 und R_3 bezeichnen wollen:

$$\operatorname*{Spur}_{\mathfrak{y}\to\mathfrak{x}}\{\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};0)-\mathfrak{G}_E(\mathfrak{x},\mathfrak{y};i\tau)\}=R_1(\mathfrak{x},\tau)+R_2(\mathfrak{x},\tau)+R_3(\mathfrak{x})\,,\tag{2.3}$$

mit

$$(4\pi)^2R_1(\mathfrak{x},\tau)=\lim_{\mathfrak{y}\to\mathfrak{x}}\operatorname{Spur}\int_G \mathfrak{G}_{12}^*(\mathfrak{x},\mathfrak{z};0)\,\{\mathfrak{G}_{21}^*(\mathfrak{z},\mathfrak{y};i\tau)\quad\mathfrak{G}_{21}^*(\mathfrak{z},\mathfrak{y};0)\}\,dV_{\mathfrak{z}}\,,\tag{2.4}$$

$$\begin{aligned}(4\pi)^2R_2(\mathfrak{x},\tau)&=\operatorname{Spur}\int_G \mathfrak{G}_{12}^*(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};i\tau)\,dV_{\mathfrak{z}}+\\&\quad+\operatorname{Spur}\int_G \mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{G}_{21}(\mathfrak{z},\mathfrak{x};i\tau)\,dV_{\mathfrak{z}}\end{aligned}\tag{2.5}$$

und

$$\begin{aligned}-(4\pi)^2R_3(\mathfrak{x})&=\operatorname{Spur}\int_G \mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{G}_{21}(\mathfrak{z},\mathfrak{x};0)\,dV_{\mathfrak{z}}+\\&\quad+\operatorname{Spur}\int_G \mathfrak{G}_{12}^*(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};0)\,dV_{\mathfrak{z}}\,.\end{aligned}\tag{2.6}$$

Wir bemerken, daß $R_3(\mathfrak{x})$ — wie schon in der Schreibweise angedeutet — nicht von τ abhängt. Daher diskutieren wir zunächst das Verhalten von R_1 und R_2.

Lemma 1. *Für* $\mathfrak{x} \in G$, $r_{\mathfrak{x}} = \inf\limits_{\mathfrak{p} \in F} |\mathfrak{p} - \mathfrak{x}| \geqq \alpha > 0$[1] *und* $\tau \geqq 0$ *ist*

$$R_1(\mathfrak{x}, \tau) = \frac{\tau}{2\pi} + C_1(\mathfrak{x}) + O\left(\frac{1}{\alpha} e^{-\alpha\tau}\right). \tag{2.7}$$

Die Funktion $C_1(\mathfrak{x})$ *hängt dabei nicht von* τ *ab und genügt der Abschätzung* $|C_1(\mathfrak{x})| = O(1/\alpha)$.

Beweis. Nach Gl. (1.4) ist

$$\operatorname{Spur} \mathfrak{G}_{12}^*(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}^*(\mathfrak{z}, \mathfrak{y}; i\tau) = -2 \nabla_{\mathfrak{z}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|} \cdot \nabla_{\mathfrak{z}} \Phi(\mathfrak{z}, \mathfrak{y}), \tag{2.8}$$

und daher

$$\frac{1}{(4\pi)^2} \operatorname{Spur} \int\limits_G \mathfrak{G}_{12}^*(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}^*(\mathfrak{z}, \mathfrak{y}; i\tau)\, dV_{\mathfrak{z}} = -\frac{1}{8\pi^2} \int\limits_G \nabla_{\mathfrak{z}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|} \nabla_{\mathfrak{z}} \Phi(\mathfrak{z}, \mathfrak{y})\, dV_{\mathfrak{z}}. \tag{2.9}$$

Den letzten Ausdruck formen wir mit Hilfe des Gaußschen Integralsatzes um und erhalten

$$\int\limits_G \nabla_{\mathfrak{z}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|} \cdot \nabla_{\mathfrak{z}} \Phi(\mathfrak{z}, \mathfrak{y})\, dV_{\mathfrak{z}} = 4\pi \Phi(\mathfrak{x}, \mathfrak{y}) + \int\limits_F \Phi(\mathfrak{z}, \mathfrak{y}) \frac{\partial}{\partial n_{\mathfrak{z}}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|}\, dF_{\mathfrak{z}}, \tag{2.10}$$

folglich

$$R_1(\mathfrak{x}, \tau) = \frac{\tau}{2\pi} + \frac{1}{8\pi^2} \int\limits_F \left(\frac{1}{|\mathfrak{z} - \mathfrak{x}|} - \Phi(\mathfrak{z}, \mathfrak{x})\right) \frac{\partial}{\partial n_{\mathfrak{z}}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|}\, dF_{\mathfrak{z}}. \tag{2.11}$$

Für alle Punkte $\mathfrak{x} \in G$ mit $r_{\mathfrak{x}} \geqq \alpha > 0$ und für alle $\tau \geqq 0$ ist

$$\int\limits_F \frac{e^{-\tau|\mathfrak{x} - \mathfrak{z}|}}{|\mathfrak{x} - \mathfrak{z}|} \frac{\partial}{\partial n_{\mathfrak{z}}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|}\, dF_{\mathfrak{z}} = O\left(\frac{e^{-\alpha\tau}}{\alpha} \int\limits_F \left| \frac{\partial}{\partial n_{\mathfrak{z}}} \frac{1}{|\mathfrak{z} - \mathfrak{x}|} \right| dF_{\mathfrak{z}}\right) = O\left(\frac{e^{-\alpha\tau}}{\alpha}\right). \tag{2.12}$$

Ferner ist

$$\int\limits_F \frac{1}{|\mathfrak{x} - \mathfrak{z}|} \frac{\partial}{\partial n_{\mathfrak{z}}} \frac{1}{|\mathfrak{x} - \mathfrak{z}|}\, dF_{\mathfrak{z}} = C_1(\mathfrak{x}) \tag{2.13}$$

eine von τ unabhängige Funktion von $\mathfrak{x}$, für die entsprechend die Abschätzung $|C_1(\mathfrak{x})| = O(1/\alpha)$ gilt. Damit ist Lemma 1 bewiesen.

Wir zeigen weiterhin

Lemma 2. *Für* $\mathfrak{x} \in G$, $r_{\mathfrak{x}} \geqq \alpha > 0$ *und* $\tau \geqq 0$ *ist*

$$R_2(\mathfrak{x}, \tau) = \int\limits_0^{\alpha/2} g(r, \mathfrak{x})\, e^{-\tau r} (1 + \tau r)\, dr + O\left(\frac{1}{\alpha} e^{-\frac{\alpha}{3}\tau}\right). \tag{2.14}$$

Die Funktion $g(r, \mathfrak{x})$ *ist dabei unabhängig von* τ, *bei festem* $\mathfrak{x}$ *als Funktion von* r *im Intervall* $0 \leqq r \leqq \alpha/2$ *stetig, und genügt dort gleichmäßig der Abschätzung* $O(1/\alpha^2)$.

Beweis. a) Nach Gl. (1.8) und der Symmetrierelation Gl. (1.12) erhalten wir den Tensor $\mathfrak{K}_{21}(\mathfrak{z}, \mathfrak{x}; i\tau)$ mit Hilfe eines Flächenfeldes $\mathfrak{v}(\mathfrak{p}, \mathfrak{z})$ $(\mathfrak{p} \in F)$ gemäß[2]

$$\mathfrak{K}_{\widetilde{21}}(\mathfrak{z}, \mathfrak{x}; i\tau)\, \mathfrak{a} = -\mathfrak{K}_{12}(\mathfrak{x}, \mathfrak{z}; i\tau)\, \mathfrak{a} = \frac{1}{4\pi} \int\limits_F \mathfrak{v}(\mathfrak{p}, \mathfrak{z}) \times \nabla_{\mathfrak{p}} \Phi(\mathfrak{p}, \mathfrak{x})\, dF_{\mathfrak{p}}, \tag{2.15}$$

---

[1] Es ist also $r_{\mathfrak{x}}$ der Abstand des Punktes $\mathfrak{x}$ von der Randfläche F.

[2] Wir wollen mit $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ Punkte aus G, mit $\mathfrak{p}$, $\mathfrak{q}$ Punkte auf F und mit $\mathfrak{v}$, $\mathfrak{w}$ Tangentialfelder an F bezeichnen.

wobei $\mathfrak{v}$ die eindeutig bestimmte Lösung der Integralgleichung

$$\mathfrak{v} - M\mathfrak{v} = \mathfrak{w}(\mathfrak{p},\mathfrak{z}) = 2\mathfrak{n}(\mathfrak{p}) \times [\nabla_{\mathfrak{p}} \Phi(\mathfrak{p},\mathfrak{z}) \times \mathfrak{a}] \tag{2.16}$$

mit

$$M\mathfrak{v} = \frac{1}{2\pi} \int\limits_F \mathfrak{n}(\mathfrak{p}) \times \big(\mathfrak{v}(\mathfrak{q},\mathfrak{z}) \times \nabla_{\mathfrak{q}} \Phi(\mathfrak{q},\mathfrak{p})\big)\, dF_{\mathfrak{q}} \tag{2.17}$$

ist.

b) Das Flächenfeld $\mathfrak{w}(\mathfrak{p},\mathfrak{z})$ wird singulär für $\mathfrak{z} \to \mathfrak{p}$, und zwar gilt

$$|\mathfrak{w}(\mathfrak{p},\mathfrak{z})| = O\big(|\nabla_{\mathfrak{p}} \Phi(\mathfrak{p},\mathfrak{z})|\big) = O\Big(\frac{1}{|\mathfrak{p}-\mathfrak{z}|^2}\Big) \tag{2.18}$$

und

$$\Big| \int\limits_{|\mathfrak{z}-\mathfrak{p}| \leqq \varrho} \mathfrak{w}(\mathfrak{p},\mathfrak{z})\, dF_{\mathfrak{p}} \Big| = O(1) \qquad \text{(unabhängig von } \varrho \text{ und } \mathfrak{z}\text{)}. \tag{2.19}$$

Beide Abschätzungen gelten gleichmäßig bezüglich $\tau \geqq 0$. Wir können daher ein Ergebnis einer früheren Arbeit [*9*] verwenden (Lemma 6 und 8 dort), nach dem

$$\begin{aligned} |M\mathfrak{w}| &= O\Big(\frac{1}{|\mathfrak{p}-\mathfrak{z}|}\Big) \\ |M^2\mathfrak{w}| &= O\Big(\log\frac{1}{|\mathfrak{p}-\mathfrak{z}|}\Big) \\ |M^3\mathfrak{w}| &= O(1) \end{aligned} \tag{2.20}$$

ist, und zwar ebenfalls gleichmäßig bezüglich aller $\tau \geqq 0$.

c) Führen wir jetzt das Flächenfeld

$$\mathfrak{v}^* = \mathfrak{v} - \mathfrak{w} - M\mathfrak{w} - M^2\mathfrak{w} \tag{2.21}$$

ein, so erfüllt $\mathfrak{v}^*$ die Integralgleichung

$$\mathfrak{v}^* - M\mathfrak{v}^* = M^3\mathfrak{w} = \mathfrak{w}^*. \tag{2.22}$$

Nun konvergiert die Neumannsche Reihe dieser Integralgleichung[1] für genügend große τ, denn es ist mit $r = |\mathfrak{p}-\mathfrak{q}|$

$$\begin{aligned} |M_\tau \mathfrak{v}^*| &\leqq \frac{1}{2\pi} \int\limits_F \Big|\mathfrak{n}(\mathfrak{p}) \times \Big(\mathfrak{v}^*(\mathfrak{q},\mathfrak{z}) \times \nabla_{\mathfrak{q}} \frac{1}{r}\Big)\Big| e^{-\tau r}(1+\tau r)\, dF_{\mathfrak{q}} \\ &= O\Big(\int\limits_F \frac{1}{r} e^{-\tau r}(1+\tau r)\, dF_{\mathfrak{q}}\Big) \cdot \|\mathfrak{v}^*\| \\ &= \|\mathfrak{v}^*\| \cdot O\Big(\frac{1}{\tau}\Big). \end{aligned} \tag{2.23}$$

Es gibt daher eine nur vom Gebiet G abhängige Konstante τ_0, so daß

$$\|M_\tau \mathfrak{v}^*\| \leqq \tfrac{1}{2} \|\mathfrak{v}^*\|$$

und somit nach Gl. (22)

$$\|\mathfrak{v}^*\| \leqq 2\|\mathfrak{w}^*\| \tag{2.24}$$

ist für alle $\tau \geqq \tau_0$.

---

[1] Wir bezeichnen die Abhängigkeit des Operators M von τ durch M_τ.

d) Für alle τ mit $0 \leqq \tau \leqq \tau_0$ ist die Integralgleichung (24) eindeutig lösbar. Für $\tau = 0$ ergibt sich dies aus [9], Lemma 5. Der Operator M_τ hängt außerdem stetig von τ ab, d.h. es ist

$$\|(M_{\tau_1} - M_{\tau_2})\,\mathfrak{v}^*\| \leqq D_1\,|\tau_1 - \tau_2| \cdot \|\mathfrak{v}^*\|, \tag{2.25}$$

mit einer nur vom Gebiet G abhängigen Konstanten D_1. Es gibt daher eine Konstante D_2, so daß für alle τ mit $0 \leqq \tau \leqq \tau_0$

$$\|\mathfrak{v}^*\| \leqq D_2 \cdot \|\mathfrak{w}^*\| \tag{2.26}$$

ist. Andernfalls nämlich gäbe es Folgen τ_n $(0 \leqq \tau_n \leqq \tau_0)$,

$$\mathfrak{v}_n^* \quad (\|\mathfrak{v}_n^*\| = 1) \quad \text{und} \quad \mathfrak{w}_n^*$$

mit $\lim\limits_{n\to\infty} \|\mathfrak{w}_n^*\| = 0$ und

$$\mathfrak{v}_n^* - M_{\tau_n}\,\mathfrak{v}_n^* = \mathfrak{w}_n^*. \tag{2.27}$$

Wir können (gegebenenfalls nach Bildung von Teilfolgen) annehmen, daß $\lim\limits_{n\to\infty} \tau_n = \tau^*$ ist $(0 \leqq \tau^* \leqq \tau_0)$. Dann wäre

$$\mathfrak{v}_n^* - M_{\tau^*}\,\mathfrak{v}_n^* = \mathfrak{w}_n^* + (M_{\tau_n} - M_{\tau^*})\,\mathfrak{v}_n^*. \tag{2.28}$$

Da die Integralgleichung für $\tau = \tau^*$ eindeutig auflösbar ist, gilt mit einer Konstanten D_3

$$\begin{aligned} \|\mathfrak{v}_n^*\| &\leqq D_3\,\|\mathfrak{w}_n^* + (M_{\tau_n} - M_{\tau^*})\,\mathfrak{v}_n^*\| \\ &\leqq D_3\,\|\mathfrak{w}_n^*\| + D_3\,D_1 \cdot |\tau_n - \tau^*|. \end{aligned} \tag{2.29}$$

Es wäre demnach $\lim\limits_{n\to\infty} \|\mathfrak{v}_n^*\| = 0$ im Widerspruch zur Annahme $\|\mathfrak{v}_n^*\| = 1$.

e) Setzt man $D = \operatorname{Max}(D_2, 2)$, so ist

$$\|\mathfrak{v}^*(\mathfrak{p}, \mathfrak{z})\| \leqq D \cdot \|\mathfrak{w}^*\| = D \cdot \|M^3\,\mathfrak{w}\| = O(1) \tag{2.30}$$

für alle $\mathfrak{z}$ aus dem abgeschlossenen Bereich $\bar{G}$ und für alle $\tau \geqq 0$. Es folgt also nach Gl. (20) und (21)

$$|\mathfrak{v}| \leqq |\mathfrak{w}| + |M\,\mathfrak{w}| + |M^2\,\mathfrak{w}| + O(1),$$

oder

$$|\mathfrak{v}(\mathfrak{p}, \mathfrak{z})| = O\left(|\mathfrak{p} - \mathfrak{z}|^{-2}\right) \tag{2.31}$$

gleichmäßig für alle $\tau \geqq 0$.

f) Aus Gl. (15) und (31) ergibt sich als Abschätzung[1] für den Tensor $\mathfrak{K}_{21}$

$$\begin{aligned} |\mathfrak{K}_{21}(\mathfrak{z}, \mathfrak{x}; i\tau)| &= O\left(\int\limits_F \frac{e^{-\tau|\mathfrak{p}-\mathfrak{x}|}(1 + \tau|\mathfrak{p}-\mathfrak{x}|)}{|\mathfrak{p}-\mathfrak{x}|^2 \cdot |\mathfrak{p}-\mathfrak{z}|^2}\,dF_{\mathfrak{p}}\right) \\ &= O\left(e^{-\frac{\alpha}{2}\tau} \int\limits_F |\mathfrak{p} - \mathfrak{x}|^{-2} \cdot |\mathfrak{p} - \mathfrak{z}|^{-2}\,dF_{\mathfrak{p}}\right) \end{aligned} \tag{2.32}$$

für $r_{\mathfrak{x}} \geqq \alpha > 0$ und alle $\tau \geqq 0$.

---

[1] Unter $|\mathfrak{K}_{21}|$ wollen wir die Summe der Absolutbeträge der einzelnen Komponenten von $\mathfrak{K}_{21}$ verstehen.

Wir erhalten nunmehr für den ersten Summanden in Gl. (5)

$$\begin{aligned}&\left|\frac{1}{(4\pi)^2}\,\mathrm{Spur}\int\limits_G \mathfrak{G}^*_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};i\tau)\,dV_{\mathfrak{z}}\right|\\ &\qquad=O\left(e^{-\frac{\alpha}{2}\tau}\int\limits_G\frac{1}{|\mathfrak{x}-\mathfrak{z}|^2}\int\limits_F\frac{1}{|\mathfrak{p}-\mathfrak{x}|^2\cdot|\mathfrak{p}-\mathfrak{z}|^2}\,dF_{\mathfrak{p}}\,dV_{\mathfrak{z}}\right)\\ &\qquad=O\left(e^{-\frac{\alpha}{2}\tau}\int\limits_F\frac{1}{|\mathfrak{p}-\mathfrak{x}|^3}\,dF_{\mathfrak{p}}\right)\\ &\qquad=O\left(\frac{1}{\alpha}\,e^{-\frac{\alpha}{2}\tau}\right).\end{aligned}\tag{2.33}$$

g) Es bleibt noch der zweite Summand in Gl. (5) zu betrachten:

$$\begin{aligned}&\frac{1}{(4\pi)^2}\,\mathrm{Spur}\int\limits_G \mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{G}_{21}(\mathfrak{z},\mathfrak{x};i\tau)\,dV_{\mathfrak{z}}\\ &\qquad=\frac{1}{(4\pi)^2}\,\mathrm{Spur}\left\{\int\limits_G\mathfrak{K}_{12}\,\mathfrak{G}^*_{21}\,dV_{\mathfrak{z}}+\int\limits_G\mathfrak{K}_{12}\,\mathfrak{K}_{21}\,dV_{\mathfrak{z}}\right\}.\end{aligned}\tag{2.34}$$

Für den Tensor $\mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)$ gilt die Abschätzung

$$|\mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)|=O(R^{-2}_{\mathfrak{x}\mathfrak{z}}),\tag{2.35}$$

wobei $R_{\mathfrak{x}\mathfrak{z}}=\inf\limits_{\mathfrak{p}\in F}(|\mathfrak{p}-\mathfrak{x}|+|\mathfrak{p}-\mathfrak{z}|)$ der „Lichtweg“ zwischen $\mathfrak{x}$ und $\mathfrak{z}$ ist ([*9*], Satz 6). Erst recht ist daher $|\mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)|=O\left(\frac{1}{|\mathfrak{x}-\mathfrak{z}|^2}\right)$ und wir erhalten mithin für den zweiten Anteil auf der rechten Seite von Gl. (34) wie in Gl. (33) die Abschätzung $O\left(\frac{1}{\alpha}\,e^{-\frac{\alpha}{2}\tau}\right)$. Bei dem ersten Anteil zerlegen wir das Integrationsgebiet in zwei Teile, nämlich in die Kugel $G_1=\left\{\mathfrak{z},\,|\mathfrak{x}-\mathfrak{z}|\leqq\frac{\alpha}{2}\right\}$ und das Restgebiet $G_2=G-G_1$.

Im Restgebiet G_2 ist $|\mathfrak{x}-\mathfrak{z}|\geqq\frac{\alpha}{2}$, d.h.

$$|\mathfrak{G}^*_{21}(\mathfrak{z},\mathfrak{x};i\tau)|=O\left(e^{-\frac{\alpha}{2}\tau}(1+\tau\,|\mathfrak{x}-\mathfrak{z}|)\,\frac{1}{|\mathfrak{x}-\mathfrak{z}|^2}\right),\tag{2.36}$$

$$\begin{aligned}\left|\int\limits_{G_2}\mathfrak{K}_{12}\,\mathfrak{G}^*_{21}\,dV_{\mathfrak{z}}\right|&=O\left(e^{-\frac{\alpha}{3}\tau}\int\limits_{G_2}\frac{1}{|\mathfrak{x}-\mathfrak{z}|^4}\,dV_{\mathfrak{z}}\right)\\ &=O\left(\frac{1}{\alpha}\,e^{-\frac{\alpha}{3}\tau}\right).\end{aligned}\tag{2.37}$$

In der Kugel G_1 ist unter Benutzung von Gl. (1.4) mit $r=|\mathfrak{x}-\mathfrak{z}|$

$$\begin{aligned}&\frac{1}{(4\pi)^2}\,\mathrm{Spur}\int\limits_{G_1}\mathfrak{K}_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{G}^*_{21}(\mathfrak{z},\mathfrak{x};i\tau)\,dV_{\mathfrak{z}}\\ &\qquad=\frac{1}{(4\pi)^2}\int\limits_{r\leqq\alpha/2}\left(\sum_{i=1}^{3}\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};0)\,\mathfrak{e}_i\times\mathfrak{e}_i\right)^{\sim}(\nabla_{\mathfrak{x}}r)\,\frac{e^{-\tau r}}{r^2}\,(1+\tau r)\,dV_{\mathfrak{z}}.\end{aligned}\tag{2.38}$$

Dabei sind $\mathfrak{e}_1$, $\mathfrak{e}_2$ und $\mathfrak{e}_3$ die Einheitsvektoren in Richtung der Koordinatenachsen.

Zerlegen wir die Volumenintegration in eine Flächenintegration und eine Integration über den Kugelradius, so folgt

$$(2.39)\qquad \frac{1}{(4\pi)^2}\,\mathrm{Spur}\int\limits_{G_1}\mathfrak{K}_{12}\,\mathfrak{G}^*_{21}\,dV_{\mathfrak{z}}=\int\limits_0^{\alpha/2} g(r,\mathfrak{x})\,e^{-\tau r}(1+\tau r)\,dr$$

mit

$$(2.40)\qquad g(r,\mathfrak{x})=\frac{1}{(4\pi)^2\cdot r^2}\int\limits_{|\mathfrak{x}-\mathfrak{z}|=r}\left(\sum_{i=1}^{3}\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};0)\,\mathfrak{e}_i\times\mathfrak{e}_i\right)^{\sim}(\nabla_{\mathfrak{x}}\,r)\,dF_{\mathfrak{z}}.$$

Da $\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};0)$ in G_1 stetig ist als Funktion von $\mathfrak{z}$, so ist auch $g(r,\mathfrak{x})$ stetig im Intervall $0\leq r\leqq\alpha/2$ als Funktion von r. Überdies genügt $g(r,\mathfrak{x})$ nach Gl. (2.35) der Abschätzung

$$(2.41)\qquad |g(r,\mathfrak{x})|=O\Big(\sup_{|\mathfrak{x}-\mathfrak{z}|=r}|\mathfrak{K}_{21}(\mathfrak{z},\mathfrak{x};0)|\Big)=O\Big(\frac{1}{\alpha^2}\Big),$$

und zwar gleichmäßig für alle r mit $0\leqq r\leqq\alpha/2$. Aus Gl. (33), (37) und (39) ergibt sich die Behauptung von Lemma 2.

Wir können nunmehr die in Gl. (2.2) auftretende Reihe abschätzen. Es gilt

Satz 1. *Für* $\mathfrak{x}\in G$, $r_{\mathfrak{x}}\geqq\alpha>0$ *und* $\tau\geqq0$ *ist*

$$(2.42)\quad \tau^2\sum_{\omega_n>0}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2(\omega_n^2+\tau^2)}=\frac{\tau}{2\pi}+C(\mathfrak{x})+\int\limits_0^{\alpha/2} g(r,\mathfrak{x})\,e^{-\tau r}(1+\tau r)\,dr+O\Big(\frac{1}{\alpha}\,e^{-\frac{\alpha}{3}\tau}\Big).$$

Die Funktion $C(\mathfrak{x})$ *und* $g(r,\mathfrak{x})$ *hängen nicht von* τ *ab.* $C(\mathfrak{x})$ *genügt der Abschätzung* $O(1/\alpha)$, $g(r,\mathfrak{x})$ *ist stetig als Funktion von* r *im Intervall* $0\leq r\leqq\alpha/2$, *und es gilt* $|g(r,\mathfrak{x})|=O(1/\alpha^2)$.

Beweis. Nach Gl. (2.3) ist die in Gl. (42) linker Hand auftretende Reihe gleich $R_1(\mathfrak{x},\tau)+R_2(\mathfrak{x},\tau)+R_3(\mathfrak{x})$. Für $R_1(\mathfrak{x},\tau)$ und $R_2(\mathfrak{x},\tau)$ haben wir bereits in Lemma 1 und 2 gezeigt, daß sie die Behauptungen unseres Satzes erfüllen. Wir haben daher lediglich zu zeigen, daß die von τ nicht abhängige Funktion $R_3(\mathfrak{x})=O(1/\alpha)$ ist. Nach Gl. (2.6) und (2.35) ist

$$(2.43)\qquad R_3(\mathfrak{x})=O\Big(\int\limits_G\frac{1}{R^2_{\mathfrak{x}\mathfrak{z}}}\cdot\frac{1}{|\mathfrak{x}-\mathfrak{z}|^2}\,dV_{\mathfrak{z}}\Big),$$

wobei $R_{\mathfrak{x}\mathfrak{z}}=\inf\limits_{\mathfrak{p}\in F}(|\mathfrak{p}-\mathfrak{x}|+|\mathfrak{p}-\mathfrak{z}|)$ der Lichtweg zwischen $\mathfrak{x}$ und $\mathfrak{z}$ ist. Zerlegen wir das Integrationsgebiet wie im Beweis zu Lemma 2 (g) in die Kugel $G_1=\{\mathfrak{z},|\mathfrak{x}-\mathfrak{z}|\leq\alpha/2\}$ und das Restgebiet $G_2=G-G_1$, so ist in G_1: $R_{\mathfrak{x}\mathfrak{z}}\geqq\frac{3}{2}\alpha$, d.h.

$$(2.44)\qquad \begin{aligned}\int\limits_{G_1}\frac{1}{R^2_{\mathfrak{x}\mathfrak{z}}}\cdot\frac{1}{|\mathfrak{x}-\mathfrak{z}|^2}\,dV_{\mathfrak{z}}&=O\Big(\frac{1}{\alpha^2}\int\limits_{G_1}\frac{1}{|\mathfrak{x}-\mathfrak{z}|^2}\,dV_{\mathfrak{z}}\Big)\\&=O\Big(\frac{1}{\alpha}\Big).\end{aligned}$$

Im Restgebiet G_2 ist $|\mathfrak{x}-\mathfrak{z}| \geqq \alpha/2$, also

$$\int_{G_2} \frac{1}{R_{\mathfrak{x}\mathfrak{z}}^2 \cdot |\mathfrak{x}-\mathfrak{z}|^2}\, dV_{\mathfrak{z}} \leqq \int_{G_2} \frac{1}{|\mathfrak{x}-\mathfrak{z}|^4}\, dV_{\mathfrak{z}} = O\left(\frac{1}{\alpha}\right). \tag{2.45}$$

Insgesamt ist also

$$|R_3(\mathfrak{x})| = O\left(\frac{1}{\alpha}\right), \tag{2.46}$$

und Satz 1 ist damit bewiesen.

Zum Abschluß dieses Paragraphen beweisen wir eine zu Satz 1 analoge Abschätzung der Bilinearentwicklung Gl. (2.1) für $\mathfrak{x} \neq \mathfrak{y}$. Wir nehmen dabei an, daß $\mathfrak{x} \in G$, $\mathfrak{y} \in G$ und $\mathfrak{x} \neq \mathfrak{y}$ ist, und setzen

$$r_{\mathfrak{y}} = \alpha > 0, \qquad |\mathfrak{x}-\mathfrak{y}| = \beta > 0, \qquad \operatorname{Min}(\alpha, \beta) = \gamma > 0.$$

Wir bezeichnen die Komponenten des Vektors $\mathfrak{E}_n(\mathfrak{x})$ mit $E_n^1(\mathfrak{x})$, $E_n^2(\mathfrak{x})$, $E_n^3(\mathfrak{x})$. Dann gilt

Satz 2. *Für $\gamma > 0$ und $\tau \geqq 0$ ist*

$$\tau^2 \sum_{\omega_n > 0} \frac{\overline{E}_n^k(\mathfrak{x})\, \overline{E}_n^l(\mathfrak{y})}{\omega_n^2(\omega_n^2+\tau^2)} = C_{kl}(\mathfrak{x}, \mathfrak{y}) + \int_0^{\gamma/2} g_{kl}(r; \mathfrak{x}, \mathfrak{y})\, e^{-\tau r}(1+\tau r)\, dr + O\left(e^{-\frac{\gamma}{2}\tau}\right). \tag{2.47}$$

Dabei hängen $C_{kl}(\mathfrak{x}, \mathfrak{y})$ und $g_{kl}(r; \mathfrak{x}, \mathfrak{y})$ nicht von τ ab. Die Funktionen $g_{kl}(r; \mathfrak{x}, \mathfrak{y})$ sind stetig im Intervall $0 \leqq r \leqq \gamma/2$ $(k, l = 1, 2, 3)$.

Beweis. Es ist nach Gl. (2.1) und Gl. (1.14)

$$\tau^2 \sum_{\omega_n > 0} \frac{E_n^k(\mathfrak{x})\, \overline{E}_n^l(\mathfrak{y})}{\omega_n^2(\omega_n^2+\tau^2)} = \mathfrak{e}_k^{\sim}\left(\mathfrak{G}_E(\mathfrak{x}, \mathfrak{y}; 0) - \mathfrak{G}_E(\mathfrak{x}, \mathfrak{y}; i\tau)\right) \mathfrak{e}_l, \tag{2.48}$$

d.h.

$$\begin{aligned} \tau^2 \sum_{\omega_n > 0} \frac{E_n^k(\mathfrak{x})\, \overline{E}_n^l(\mathfrak{y})}{\omega_n^2(\omega_n^2+\tau^2)} = &- \frac{1}{(4\pi)^2} \int_G \mathfrak{e}_k^{\sim}\, \mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}(\mathfrak{z}, \mathfrak{y}; 0)\, \mathfrak{e}_l\, dV_{\mathfrak{z}} + \\ &+ \frac{1}{(4\pi)^2} \int_G \mathfrak{e}_k^{\sim}\, \mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}(\mathfrak{z}, \mathfrak{y}; i\tau)\, \mathfrak{e}_l\, dV_{\mathfrak{z}}. \end{aligned} \tag{2.49}$$

Nun setzen wir

$$C_{kl}(\mathfrak{x}, \mathfrak{y}) = -\frac{1}{(4\pi)^2} \int_G \mathfrak{e}_k^{\sim}\, \mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}(\mathfrak{z}, \mathfrak{y}; 0)\, \mathfrak{e}_l\, dV_{\mathfrak{z}}. \tag{2.50}$$

Ferner ist nach Gl. (32)

$$\begin{aligned} \frac{1}{(4\pi)^2} \int_G \mathfrak{e}_k^{\sim}\, \mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{K}_{21}(\mathfrak{z}, \mathfrak{y}; i\tau)\, \mathfrak{e}_l\, dV_{\mathfrak{z}} &= O\left(e^{-\frac{\alpha}{2}\tau} \int_G |\mathfrak{x}-\mathfrak{z}|^{-2} \log r_{\mathfrak{z}}^{-1}\, dV_{\mathfrak{z}}\right) \\ &= O\left(e^{-\frac{\alpha}{2}\tau}\right) = O\left(e^{-\frac{\gamma}{2}\tau}\right). \end{aligned} \tag{2.51}$$

In dem verbleibenden Anteil

$$\frac{1}{(4\pi)^2} \int_G \mathfrak{e}_k^{\sim}\, \mathfrak{G}_{12}(\mathfrak{x}, \mathfrak{z}; 0)\, \mathfrak{G}_{21}^*(\mathfrak{z}, \mathfrak{y}; i\tau)\, \mathfrak{e}_l\, dV_{\mathfrak{z}}$$

zerlegen wir das Integrationsgebiet wieder in die Kugel $G_1=\{\mathfrak{z},\,|\mathfrak{y}-\mathfrak{z}|\leqq\gamma/2\}$ und das Restgebiet $G_2=G-G_1$, und erhalten analog zu Gl. (40)

$$\frac{1}{(4\pi)^2}\int\limits_{G_1}\mathfrak{e}_k^{\sim}\,\mathfrak{G}_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{G}_{21}^{*}(\mathfrak{z},\mathfrak{y};i\tau)\,\mathfrak{e}_l\,dV=\int\limits_0^{\gamma/2}g_{kl}(r;\mathfrak{x},\mathfrak{y})\,e^{-\tau r}(1+\tau r)\,dr \tag{2.52}$$

mit

$$g_{kl}(r;\mathfrak{x},\mathfrak{y})=\frac{1}{(4\pi)^2}\int\limits_{|\mathfrak{y}-\mathfrak{z}|\;r}\left(\mathfrak{G}_{21}(\mathfrak{z},\mathfrak{x};0)\,\mathfrak{e}_k\times\mathfrak{e}_l\right)\nabla_{\mathfrak{y}}|\mathfrak{z}-\mathfrak{y}|\,dF_{\mathfrak{z}}. \tag{2.53}$$

Schließlich ist im Restgebiet G_2: $|\mathfrak{z}-\mathfrak{y}|\geqq\gamma/2$, d.h.

$$\left|\mathfrak{G}_{21}^{*}(\mathfrak{z},\mathfrak{y};i\tau)\,\mathfrak{e}_l\right|=O\left(e^{-\frac{\gamma}{2}\tau}\right) \tag{2.54}$$

daher

$$\frac{1}{(4\pi)^2}\int\limits_{G_2}\mathfrak{e}_k^{\sim}\,\mathfrak{G}_{12}(\mathfrak{x},\mathfrak{z};0)\,\mathfrak{G}_{21}^{*}(\mathfrak{z},\mathfrak{y};i\tau)\,\mathfrak{e}_l\,dV_{\mathfrak{z}}=O\left(e^{-\frac{\gamma}{2}\tau}\right). \tag{2.55}$$

Damit sind alle Ausdrücke abgeschätzt, die in der Bilinearentwicklung auftreten. Aus Gl. (50) bis (55) ergibt sich die Behauptung für Satz 2.

§ 3. Anwendung eines Tauberschen Satzes

Wir stellen zunächst einige bekannte Formeln über uneigentliche Integrale zusammen. Mit

$$\mu(r,x)=1+\frac{2}{\pi}\sin r\sqrt{x}-\frac{2}{\pi}\int\limits_0^{r\sqrt{x}}\frac{\sin\xi}{\xi}\,d\xi \tag{3.1}$$

gilt

$$\lambda\int\limits_0^{\infty}\frac{\mu(r,x)}{(\lambda+x)^2}\,dx=e^{-r\sqrt{\lambda}}(1+r\sqrt{\lambda}) \tag{3.2}$$

und

$$|\mu(r,x)|\leqq 2\left(1+\frac{1}{\pi}\right)\quad\text{für alle } r,\ x\geqq 0. \tag{3.3}$$

Ferner ist

$$\lambda\int\limits_0^{\infty}\frac{\sqrt{x}}{(\lambda+x)^2}\,dx=\frac{\pi}{2}\sqrt{\lambda} \tag{3.4}$$

und

$$\frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin s\sqrt{\lambda}}{s(\lambda+u)^2}\,d\lambda=\frac{1}{\sqrt{u}}\,e^{-|s|\sqrt{u}},\qquad u>0. \tag{3.5}$$

Es sei $\mathfrak{x}\in G$, $r_{\mathfrak{x}}\geqq\alpha>0$, und x eine nicht-negative reelle Zahl. Wir führen folgende Bezeichnungen ein:

$$h(x,\mathfrak{x})=\int\limits_0^{\alpha/2}g(r,\mathfrak{x})\,\mu(r,x)\,dr, \tag{3.6}$$

und

$$A(x,\mathfrak{x})=\begin{cases}\displaystyle\sum_{0<\omega_n\leqq\sqrt{x}}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2}-\frac{\sqrt{x}}{\pi^2}-C(\mathfrak{x})-h(x,\mathfrak{x}) & \text{für}\quad x>0\\ 0 & \text{für}\quad x=0.\end{cases} \tag{3.7}$$

Dabei sind $C(\mathfrak{x})$ und $g(r, \mathfrak{x})$ die bereits in Satz 1 auftretenden Funktionen. Ferner bilden wir

$$B(\lambda, \mathfrak{x}) = \lambda \int_0^\infty \frac{A(x, \mathfrak{x})}{(\lambda + x)^2}\, dx \tag{3.8}$$

und

$$\Theta_\delta(x, \mathfrak{x}) = \sum_{0 < \omega_n \leqq \sqrt{x}} \frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^{2\delta}}. \tag{3.9}$$

Mit diesen Bezeichnungen gilt

$$\lambda \sum_{n=1}^\infty \frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2(\omega_n^2 + \lambda)} = \lambda \int_0^\infty \frac{d\Theta_1(x, \mathfrak{x})}{\lambda + x}, \tag{3.10}$$

bzw. nach partieller Integration

$$\lambda \sum_{n=1}^\infty \frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2(\omega_n^2 + \lambda)} = \lambda \int_0^\infty \frac{\Theta_1(x, \mathfrak{x})}{(\lambda + x)^2}\, dx. \tag{3.11}$$

Aus der Konvergenz der Stieltjes-Transformation Gl. (10) folgt nämlich bereits $\Theta_1(x, \mathfrak{x}) = o(x)$, so daß die partielle Integration statthaft ist.

Wir beweisen nun

Lemma 3. *Es ist für* $x \geqq 0$ *bzw.* $\lambda \geqq 0$ *und* $r_{\mathfrak{x}} \geqq \alpha > 0$

$$|h(x, \mathfrak{x})| = O\left(\frac{1}{\alpha}\right), \tag{3.12}$$

$$|A(x, \mathfrak{x})| = O(\sqrt{x}) + O\left(\frac{1}{\alpha}\right) \tag{3.13}$$

und

$$|B(\lambda, \mathfrak{x})| = O\left(\frac{1}{\alpha} e^{-\frac{\alpha}{3}\sqrt{\lambda}}\right). \tag{3.14}$$

Beweis. Die Abschätzung Gl. (12) folgt aus Satz 1 und Gl. (3.3). Ferner ergibt sich nach Satz 1 und Gl. (3.11)

$$\lambda \int_0^\infty \frac{\Theta_1(x, \mathfrak{x})}{(\lambda + x)^2}\, dx = O(\sqrt{\lambda}) + O\left(\frac{1}{\alpha}\right). \tag{3.15}$$

Da $\Theta_1(x, \mathfrak{x})$ monoton wächst, ist

$$\Theta_1(\lambda, \mathfrak{x}) = \Theta_1(\lambda, \mathfrak{x}) \cdot 2\lambda \int_\lambda^\infty \frac{dx}{(x + \lambda)^2} \leqq 2\lambda \int_0^\infty \frac{\Theta_1(x, \mathfrak{x})}{(x + \lambda)^2}\, dx = O(\sqrt{\lambda}) + O\left(\frac{1}{\alpha}\right). \tag{3.16}$$

Aus Gl. (7), (9) und (16) folgt Gl. (13). Nun ist nach Gl. (7)

$$\begin{aligned} \lambda \int_0^\infty \frac{A(x, \mathfrak{x})}{(x + \lambda)^2}\, dx = \lambda \int_0^\infty \frac{\Theta_1(x, \mathfrak{x})}{(x + \lambda)^2}\, dx - \frac{\lambda}{\pi^2} \int_0^\infty \frac{\sqrt{x}}{(x + \lambda)^2}\, dx - \\ - \lambda \cdot C(\mathfrak{x}) \int_0^\infty \frac{dx}{(x + \lambda)^2} - \lambda \int_0^\infty \frac{h(x, \mathfrak{x})}{(x + \lambda)^2}\, dx. \end{aligned} \tag{3.17}$$

Benutzen wir daher die Formeln (2), (4) und (6), so erhalten wir

$$
\begin{aligned}
\lambda \int_0^\infty \frac{A(x,\mathfrak{x})}{(x+\lambda)^2}\,dx
&= \lambda \sum_{n=1}^\infty \frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2(\omega_n^2+\lambda)} - \frac{\sqrt{\lambda}}{2\pi} - C(\mathfrak{x}) - \lambda \int_0^{\alpha/2} g(r,\mathfrak{x}) \int_0^\infty \frac{\mu(r,x)}{(x+\lambda)^2}\,dx\,dr \\
&= \lambda \sum_{n=1}^\infty \frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2(\omega_n^2+\lambda)} - \frac{\sqrt{\lambda}}{2\pi} - C(\mathfrak{x}) - \int_0^{\alpha/2} g(r,\mathfrak{x})\, e^{-r\sqrt{\lambda}}(1+r\sqrt{\lambda})\,dr.
\end{aligned}
\tag{3.18}
$$

Nach Satz 1 ist mithin

$$\lambda \int_0^\infty \frac{A(x,\mathfrak{x})}{(x+\lambda)^2}\,dx = O\left(\frac{1}{\alpha}\, e^{-\frac{\alpha}{3}\sqrt{\lambda}}\right). \tag{3.19}$$

Andererseits ist

$$\lambda \int_0^\infty \frac{A(x,\mathfrak{x})}{(x+\lambda)^2}\,dx = B(\lambda,\mathfrak{x}). \tag{3.20}$$

Aus Gl. (19) und (20) folgt Gl. (14). Damit sind sämtliche Behauptungen von Lemma 3 bewiesen.

Weiterhin gilt

Lemma 4.

$$\varphi(z,\mathfrak{x}) = \int_0^\infty e^{-zv} A(v^2,\mathfrak{x})\,dv \tag{3.21}$$

konvergiert für $Re(z) > 0$ *und ist dort holomorph. Die Funktion* $\varphi(z,\mathfrak{x})$ *ist holomorph fortsetzbar in den Kreis* $|z| < \alpha/3$ *und ist in* $|z| \leqq \alpha/4$ *beschränkt.*

Beweis. Die erste Behauptung des Lemmas folgt unmittelbar aus Gl. (13). Ferner ist

$$\varphi(z,\mathfrak{x}) = \frac{1}{2} \int_0^\infty \frac{e^{-z\sqrt{u}}}{\sqrt{u}} A(u,\mathfrak{x})\,du. \tag{3.22}$$

Für reelle $z > 0$ gilt daher unter Benutzung von Gl. (5)

$$\varphi(z,\mathfrak{x}) = \frac{1}{\pi} \int_0^\infty \left(\int_0^\infty A(u,\mathfrak{x}) \frac{\sin z\sqrt{\lambda}}{z(\lambda+u)^2}\,d\lambda \right) du. \tag{3.23}$$

Wir dürfen die Reihenfolge der Integrationen vertauschen und erhalten nach Gl. (20)

$$\varphi(z,\mathfrak{x}) = \frac{2}{\pi} \int_0^\infty B(\lambda,\mathfrak{x}) \frac{\sin z\sqrt{\lambda}}{z\cdot\lambda}\,d\lambda. \tag{3.24}$$

Nun ist aber für komplexe z und reelle positive λ

$$\left| \frac{\sin z\sqrt{\lambda}}{z\lambda} \right| \leqq \frac{1}{\sqrt{\lambda}}\, e^{|z|\sqrt{\lambda}}, \tag{3.25}$$

d.h.

$$(3.26)\qquad |\varphi(z,\mathfrak{x})| \leqq \frac{2}{\pi}\int\limits_0^\infty |B(\lambda,\mathfrak{x})|\,\frac{e^{|z|\sqrt{\lambda}}}{\sqrt{\lambda}}\,d\lambda = O\left(\frac{1}{\alpha}\int\limits_0^\infty e^{-\left(\frac{\alpha}{3}-|z|\right)\sqrt{\lambda}}\,\frac{d\lambda}{\sqrt{\lambda}}\right).$$

Die Integraldarstellungen Gl. (21) und (24) stellen für positive reelle z dieselbe Funktion dar. Das Integral Gl. (24) konvergiert aber für alle $|z|<\alpha/3$ und stellt dort folglich eine holomorphe Funktion $\varphi(z,\mathfrak{x})$ dar, die nach Gl. (26) in $|z|\leqq\alpha/4$ beschränkt ist. Damit ist Lemma 4 bewiesen.

Wir benutzen nunmehr folgenden Tauberschen Satz von A. E. Ingham[1].

Satz 3. *Die Funktion $A(x)$, $A(0)=0$, sei reell für reelle $x\geqq 0$ und von beschränkter Variation in jedem endlichen Intervall. Ferner gebe es eine Funktion $H(\delta)>0$ so, daß für jedes $\delta>0$*

$$(3.27)\qquad A(x'^2)-A(x^2)\geqq -H(\delta)$$

für $x\leqq x'\leqq x+\delta$, $x\geqq x_0(\delta)\geqq 0$ ist. Das uneigentliche Integral

$$(3.28)\qquad \varphi^*(z)=\int\limits_0^\infty e^{-zx}\,dA(x^2)$$

sei konvergent für $Re(z)=\sigma>0$ $(z=\sigma+it)$, und $\varphi^*(z)$ erfülle die Bedingung: $\dfrac{\varphi^*(z)-\varphi^*(0)}{z}$ ist stetig in $\sigma\geqq 0$, $-T\leqq t\leqq +T$.

Dann gilt

$$(3.29)\qquad \overline{\lim_{x\to\infty}}\,|A(x^2)-\varphi^*(0)|\leqq M\cdot H(T^{-1}),$$

wo M eine absolute Konstante ist.

In unserem Fall ist

$$(3.30)\qquad \begin{aligned} A(x'^2,\mathfrak{x})-A(x^2,\mathfrak{x}) &= \sum_{x\leqq\omega_n<x'}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2}-\frac{x'-x}{\pi^2}-\{h(x'^2,\mathfrak{x})-h(x^2,\mathfrak{x})\}\\ &\geqq -\frac{\delta}{\pi^2}-\int\limits_0^{\alpha/2} g(r,\mathfrak{x})\,\{\mu(r,x'^2)-\mu(r,x^2)\}\,dr, \end{aligned}$$

und nach Gl. (3.1)

$$(3.31)\qquad \begin{aligned} |\mu(r,x'^2)-\mu(r,x^2)| &= \frac{2}{\pi}\left|\sin r x'-\sin r x-\int\limits_{rx}^{rx'}\frac{\sin\xi}{\xi}\,d\xi\right|\\ &\leqq \frac{4r}{\pi}(x'-x)\leqq\frac{4r\delta}{\pi} \end{aligned}$$

so daß mit einer Konstanten

$$M_1(\mathfrak{x})=\frac{1}{\pi^2}+\frac{4}{\pi}\int\limits_0^{\alpha/2} r\cdot|g(r,\mathfrak{x})|\,dr$$

$$(3.32)\qquad A(x'^2,\mathfrak{x})-A(x^2,\mathfrak{x})\geqq -M_1(\mathfrak{x})\,\delta\quad\text{für}\quad x\leqq x'\leqq x+\delta,\qquad \delta\geqq 0$$

gilt. Ferner ist für $Re(z)>0$

$$(3.33)\qquad \varphi^*(z,\mathfrak{x})=\int\limits_0^\infty e^{-xz}\,dA(x^2,\mathfrak{x})=z\int\limits_0^\infty e^{-xz}A(x^2,\mathfrak{x})\,dx=z\,\varphi(z,\mathfrak{x}).$$

---

[1] A. E. Ingham [5], Theorem III dort.

Nach Lemma 4 ist also

$$\frac{\varphi^*(z,\mathfrak{x})-\varphi^*(0,\mathfrak{x})}{z}=\varphi(z,\mathfrak{x}) \tag{3.34}$$

stetig in $\sigma \geqq 0$, $-\alpha/4 \leqq t \leqq +\alpha/4$. Alle Voraussetzungen des Satzes von INGHAM sind mithin erfüllt und wir erhalten wegen $\varphi^*(0,\mathfrak{x})=0$ als Ergebnis

$$\overline{\lim_{x\to\infty}}\,|A(x^2,\mathfrak{x})| \leqq M\cdot M_1(\mathfrak{x})\cdot\frac{4}{\alpha}.$$

Die Konstante $M_1(\mathfrak{x})$ ist gleichmäßig beschränkt für alle $\mathfrak{x}\in G$. Es gilt daher

Lemma 5. *Es sei $\mathfrak{x}\in G$, $r_{\mathfrak{x}} \geqq \alpha > 0$. Dann ist*

$$\Theta_1(x,\mathfrak{x})=\sum_{0<\omega_n\leqq\sqrt{x}}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2}=\frac{\sqrt{x}}{\pi^2}+S(x,\mathfrak{x}), \tag{3.35}$$

wobei $S(x,\mathfrak{x})$ der Abschätzung $|S(x,\mathfrak{x})|=O(1/\alpha)$ genügt[1].

Wir wollen aus Lemma 5 einige Folgerungen ziehen. Zunächst beweisen wir entsprechende Formeln für $\Theta_\delta(x,\mathfrak{x})$ $(\delta \neq 1)$. Es ist

$$\Theta_\delta(x,\mathfrak{x})=\sum_{0<\omega_n\leqq\sqrt{x}}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^{2\delta}}=\int\limits_{\omega_1^2-0}^{x}\frac{d\,\Theta_1(t,\mathfrak{x})}{t^{\delta-1}}, \tag{3.36}$$

und durch partielle Integration folgt

$$\Theta(x,\mathfrak{x})=\frac{\Theta_1(x,\mathfrak{x})}{x^{\delta-1}}+(\delta-1)\int\limits_{\omega_1^2}^{x}\frac{\Theta_1(t,\mathfrak{x})}{t^\delta}\,dt. \tag{3.37}$$

Führen wir für $\Theta_1(x,\mathfrak{x})$ den in Lemma 5 gefundenen Ausdruck ein, und setzen zur Abkürzung noch

$$W_\delta(\mathfrak{x})=\begin{cases}(\delta-1)\displaystyle\int\limits_{\omega_1^2}^{\infty}\frac{S(t,\mathfrak{x})}{t^\delta}\,dt-\frac{\omega_1^{3-2\delta}(2\delta-2)}{\pi^2(3-2\delta)} & \left(\delta>1,\ \delta\neq\frac{3}{2}\right)\\ \displaystyle\frac{1}{2}\int\limits_{\omega_1^2}^{\infty}\frac{S(t,\mathfrak{x})}{t^{\frac{3}{2}}}\,dt+\frac{1}{\pi^2}(1-\log\omega_1) & \left(\delta=\frac{3}{2}\right)\\ 0 & (\delta\leqq 1)\end{cases} \tag{3.38}$$

so erhalten wir folgendes Ergebnis:

Satz 4. *Es sei $\mathfrak{x}\in G$, $r_{\mathfrak{x}}\geqq\alpha>0$. Dann ist für $x\to\infty$*

$$\Theta_\delta(x,\mathfrak{x})=\begin{cases}\dfrac{x^{\frac{3}{2}-\delta}}{\pi^2(3-2\delta)}+W_\delta(\mathfrak{x})+O\left(\dfrac{1}{\alpha\,x^{\delta-1}}\right) & \text{für}\quad \delta\neq\dfrac{3}{2}\\ \dfrac{1}{2\pi^2}\log x+W_{\frac{3}{2}}(\mathfrak{x})+O\left(\dfrac{1}{\alpha\sqrt{x}}\right) & \text{für}\quad \delta=\dfrac{3}{2}.\end{cases} \tag{3.39}$$

Insbesondere konvergiert $\sum\limits_{n=1}^{\infty}\dfrac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^{2\delta}}$ *für* $\delta>\frac{3}{2}$ *und divergiert für* $\delta\leqq\frac{3}{2}$.

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[1] Am Beispiel der Eigenschwingungen in einer Kugel läßt sich zeigen, daß dieses Ergebnis im allgemeinen nicht verbessert werden kann.

Zum Abschluß dieses Paragraphen beweisen wir

Satz 5. *Die Anzahl der Eigenfrequenzen* ω_n *im Intervall* $0<\omega_n\leqq x$ *genügt für* $x\to\infty$ *der Abschätzung*

$$\sum_{0<\omega_n\leqq x} 1 = \frac{V\cdot x^3}{3\pi^2} + O(x^2\log x). \tag{3.40}$$

Hierbei ist V das Volumen des betrachteten Gebietes G.

Beweis. Die Bilinearentwicklung Gl. (2.2) konvergiert für alle $\tau\geqq 0$, und zwar gleichmäßig im abgeschlossenen Bereich $\overline{G}$. Setzen wir

$$\lim_{x\to\infty}\Theta_2(x,\mathfrak{x}) = W_2(\mathfrak{x}),$$

so ist also $W_2(\mathfrak{x})$ in $\overline{G}$ stetig. Andererseits ist nach Satz 4

$$\Theta_2(x^2,\mathfrak{x}) = W_2(\mathfrak{x}) - \frac{1}{\pi^2 x} + O\left(\frac{1}{\alpha x^2}\right). \tag{3.41}$$

Wir integrieren diese Gleichung über G_ϱ, wobei wir zu G_ϱ alle die Punkte $\mathfrak{x}\in G$ zählen, die von F einen Abstand $\geqq\varrho$ haben: $G_\varrho=\{\mathfrak{x}\in G,\ r_{\mathfrak{x}}\geqq\varrho>0\}$. Dann ist zunächst

$$0\leqq\int_{G-G_\varrho}\Theta_2(x^2,\mathfrak{x})\,dV_{\mathfrak{x}}\leqq\int_{G-G_\varrho}W_2(\mathfrak{x})\,dV_{\mathfrak{x}} = O(\varrho), \tag{3.42}$$

und es ergibt sich wegen der Stetigkeit von $W_2(\mathfrak{x})$ in $\overline{G}$

$$\sum_{0<\omega_n\leqq x}\frac{1}{\omega_n^4} = \int_G W_2(\mathfrak{x})\,dV_{\mathfrak{x}} - \frac{V}{\pi^2 x} + O(\varrho) + O\left(\frac{\varrho}{x}\right) + O\left(\frac{1}{x^2}\log\frac{1}{\varrho}\right). \tag{3.43}$$

Setzen wir jetzt $\varrho=1/x^2$, so folgt für $x\to\infty$

$$\sum_{0<\omega_n\leqq x}\frac{1}{\omega_n^2} = W_2^* - \frac{V}{\pi^2 x} + O\left(\frac{1}{x^2}\log x\right) \tag{3.44}$$

mit

$$W_2^* = \int_G W_2(\mathfrak{x})\,dV_{\mathfrak{x}}.$$

Durch partielle Integration ergibt sich daraus

$$\sum_{0<\omega_n\leqq x} 1 = \frac{V\cdot x^3}{3\pi^2} + O(x^2\log x), \tag{3.45}$$

d.h. die Behauptung von Satz 5.

§ 4. Weitere Ergebnisse

Wir wollen in diesem Paragraphen zunächst beweisen, daß für $\mathfrak{x}\neq\mathfrak{y}$

$$\left|\sum_{0<\omega_n\leqq x}\frac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}_n^{\sim}(\mathfrak{y})}{\omega_n^2}\right| = O(1) \qquad (x\to\infty)$$

ist. Dazu wenden wir den schon zitierten Tauberschen Satz von Ingham auf die in Satz 2 gewonnene Abschätzung der Bilinearentwicklung an. Der Beweisgang ist weitgehend analog zu den Ableitungen in §3. Allerdings werden wir an einer Stelle von dem Ergebnis des Satzes 4 Gebrauch machen.

Wir wollen annehmen, daß die Punkte $\mathfrak{x}$ und $\mathfrak{y}$ $(\mathfrak{x}\in G,\ \mathfrak{y}\in G,\ \mathfrak{x}\neq\mathfrak{y})$ für das Weitere fest gewählt sind. Anschließend an Satz 2 führen wir folgende Bezeichnungen ein (vgl. die analogen Bezeichnungen in §3):

$$h_{kl}(x)=\int_0^{\gamma/2} g_{kl}(r;\mathfrak{x},\mathfrak{y})\,\mu(r,x)\,dr, \tag{4.1}$$

$$\Theta_{kl}^{\delta}(x)=\sum_{0<\omega_n\leqq\sqrt{x}}\frac{E_n^k(\mathfrak{x})\,\overline{E}_n^l(\mathfrak{y})}{\omega_n^{2\delta}}, \tag{4.2}$$

$$A_{kl}(x)=\Theta_{kl}^1(x)-C_{kl}(\mathfrak{x},\mathfrak{y})-h_{kl}(x), \tag{4.3}$$

$$B_{kl}(\lambda)=\lambda\int_0^{\infty}\frac{A_{kl}(x)}{(x+\lambda)^2}\,dx, \qquad (k,l=1,2,3). \tag{4.4}$$

Dann gilt

Lemma 6. *Es ist für* $x\geqq 0$ *bzw.* $\lambda\geqq 0$

$$h_{kl}(x)=O(1), \tag{4.5}$$

$$A_{kl}(x)=\Theta_{kl}^1(x)+O(1)=O\left(\sqrt{x}\right)+O(1), \tag{4.6}$$

$$B_{kl}(\lambda)=O\left(e^{-\frac{\gamma}{2}\sqrt{\lambda}}\right). \tag{4.7}$$

Beweis. Die erste Abschätzung Gl. (5) folgt unmittelbar aus Satz 2 und Gl. (3.1). Gl. (6) folgt mittels der Schwarzschen Ungleichung aus Satz 4, denn es ist

$$\begin{aligned}|\Theta_{kl}^1(x)|^2 &\leqq \sum_{0<\omega_n\leqq\sqrt{x}}\frac{|E_n^k(\mathfrak{x})|^2}{\omega_n^2}\cdot\sum_{0<\omega_n\leqq\sqrt{x}}\frac{|E_n^l(\mathfrak{y})|^2}{\omega_n^2}\\ &\leqq \sum_{0<\omega_n\leqq\sqrt{x}}\frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2}\cdot\sum_{0<\omega_n\leqq\sqrt{x}}\frac{|\mathfrak{E}_n(\mathfrak{y})|^2}{\omega_n^2}\\ &=O(x).\end{aligned} \tag{4.8}$$

Gl. (7) schließlich ergibt sich bei Verwendung von Satz 2 aus

$$\begin{aligned}B_{kl}(\lambda)&=\lambda\int_0^{\infty}\frac{A_{kl}(x)}{(\lambda+x)^2}\,dx\\ &=\lambda\sum_{n=1}^{\infty}\frac{E_n^k(\mathfrak{x})\,\overline{E}_n^l(\mathfrak{y})}{\omega_n^2(\omega_n^2+\lambda)}-C_{kl}(\mathfrak{x},\mathfrak{y})-\int_0^{\gamma/2} g(r;\mathfrak{x},\mathfrak{y})\,e^{-r\sqrt{\lambda}}\left(1+r\sqrt{\lambda}\right)dr\\ &=O\left(e^{-\frac{\gamma}{2}\sqrt{\lambda}}\right).\end{aligned} \tag{4.9}$$

Man vgl. hierzu die analogen Herleitungen in §3, Gl. (17) bis (20).

Die Funktionen

$$\varphi_{kl}(z)=\int_0^{\infty} e^{-zv}A_{kl}(v^2)\,dv \qquad (k,l=1,2,3) \tag{4.10}$$

sind daher ebenfalls holomorph für $Re(z)>0$ und holomorph fortsetzbar in den Kreis $|z|<\gamma/2$ und beschränkt in $|z|\leqq\gamma/3$.

Wir wollen schließlich noch zeigen, daß die Funktionen $A_{kl}(x)$ auch die restlichen Voraussetzungen des Satzes von Ingham (Satz 3) erfüllen und beweisen dazu

Lemma 7. *Es gibt positive Konstanten* M_{kl} *und* K, *so daß für jedes* $\delta > 0$

$$A_{kl}(x'^2) - A_{kl}(x^2) \geqq -(M_{kl}\delta + K) \tag{4.11}$$

ist für alle x' *mit* $0 \leqq x \leqq x' \leqq x + \delta$.

Beweis. Es ist

$$A_{kl}(x'^2) - A_{kl}(x^2) = \sum_{x < \omega_n \leqq x'} \frac{E_n^k(\mathfrak{x})\,\overline{E}_n^l(\mathfrak{y})}{\omega_n^2} - \{h_{kl}(x'^2) - h_{kl}(x^2)\}. \tag{4.12}$$

Nach der Schwarzschen Ungleichung und nach Satz 4 bzw. Gl. (3.34) erhalten wir mit einer geeigneten positiven Konstanten K

$$\begin{aligned} \left|\sum_{x < \omega_n \leqq x'} \frac{E_n^k(\mathfrak{x})\,\overline{E}_n^l(\mathfrak{y})}{\omega_n^2}\right|^2 &\leqq \sum_{x < \omega_n \leqq x'} \frac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^2} \sum_{x < \omega_n \leqq x'} \frac{|\mathfrak{E}_n(\mathfrak{y})|^2}{\omega_n^2} \\ &\leqq \left(\frac{\delta}{\pi^2} + K\right)^2 \end{aligned} \tag{4.13}$$

d.h.

$$\sum_{x < \omega_n \leqq x'} \frac{E_n^k(\mathfrak{x})\,\overline{E}_n^l(\mathfrak{y})}{\omega_n^2} \geqq -\left(\frac{\delta}{\pi^2} + K\right). \tag{4.14}$$

Ferner ist nach Gl. (3.31) und (4.1)

$$h_{kl}(x'^2) - h_{kl}(x^2) \geqq -\frac{4\delta}{\pi} \int_0^{\gamma/2} r \cdot |g_{kl}(r; \mathfrak{x}, \mathfrak{y})|\,dr, \tag{4.15}$$

mithin insgesamt

$$\begin{aligned} A_{kl}(x'^2) - A_{kl}(x^2) &\geqq -\delta\left(\frac{1}{\pi^2} + \frac{4}{\pi}\int_0^{\gamma/2} r\,|g_{kl}(r; \mathfrak{x}, \mathfrak{y})|\,dr\right) - K \\ &\geqq -(M_{kl}\delta + K), \end{aligned} \tag{4.16}$$

für $0 \leqq x \leqq x' \leqq x + \delta$ und alle $\delta \geqq 0$.

Die Voraussetzungen des Satzes von Ingham sind daher erfüllt und wir erhalten nach Satz 3

$$A_{kl}(x) = O(1) \tag{4.17}$$

bzw. nach Gl. (4.3) und (4.6)

$$\sum_{0 < \omega_n \leqq x} \frac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}_n^{\sim}(\mathfrak{y})}{\omega_n^2} = O(1), \qquad \mathfrak{x} \neq \mathfrak{y}, \quad x \to \infty. \tag{4.18}$$

Wir wollen noch untersuchen, für welche Werte von δ der Ausdruck $\Theta_{kl}^{\delta}(x)$ konvergiert für $x \to \infty$. Es ist

$$\Theta_{kl}^{\delta}(x) - \Theta_{kl}^{\delta}(y) = \int_y^x \frac{d\,\Theta_{kl}^1(t)}{t^{\delta-1}}, \tag{4.19}$$

oder nach partieller Integration

$$(4.20)\qquad \Theta^{\delta}_{kl}(x) - \Theta^{\delta}_{kl}(y) = \frac{\Theta^{1}_{kl}(x)}{x^{\delta-1}} - \frac{\Theta^{1}_{kl}(y)}{y^{\delta-1}} + (\delta-1)\int\limits_{y}^{x} \frac{\Theta^{1}_{kl}(t)}{t^{\delta}}\,d\lambda .$$

Wegen $\Theta^{1}_{kl}(x) = O(1)$ ist

$$(4.21)\qquad \left|\Theta^{\delta}_{kl}(x) - \Theta^{\delta}_{kl}(y)\right| = O\left(\frac{1}{x^{\delta-1}}\right) + O\left(\frac{1}{y^{\delta-1}}\right),$$

so daß die rechte Seite im Fall $\delta > 1$ beliebig klein gemacht werden kann für genügend große x und y. $\Theta^{\delta}_{kl}(x)$ konvergiert daher, falls $\delta > 1$ ist. Es gilt mithin

Satz 6. *Für $\mathfrak{x} \in G$, $\mathfrak{y} \in G$, $\mathfrak{x} \neq \mathfrak{y}$ und $x \to \infty$ ist*

$$\sum_{0 < \omega_n \leq x} \frac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}{}_n^{\sim}(\mathfrak{y})}{\omega_n^2} = O(1).$$

Die Reihe $\sum\limits_{n=1}^{\infty} \dfrac{\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}{}_n^{\sim}(\mathfrak{y})}{\omega_n^{2\delta}}$ *konvergiert für $\delta > 1$.*

Wir bemerken abschließend, daß wegen der Symmetrie der Greenschen Tensoren Gl. (1.12) aus der Bilinearrelation wie in Satz 4 und 6 auch für die magnetischen Eigenschwingungen gelten. Sind ε und μ jedoch von 1 verschieden, so ergeben sich aus den Substitutionen Gl. (1.2) folgende Ergebnisse:

Satz 7. *Es seien $\mathfrak{E}_n$, $\mathfrak{H}_n$ die elektrischen bzw. magnetischen Eigenschwingungen zur Eigenfrequenz ω_n der Maxwellschen Gleichungen*

$$\nabla \times \mathfrak{E} - i\,\omega\,\mu\,\mathfrak{H} = \mathfrak{o}, \qquad \nabla \times \mathfrak{H} + i\,\omega\,\varepsilon\,\mathfrak{E} = \mathfrak{o}$$

mit der Randbedingung $\mathfrak{n} \times \mathfrak{E} = \mathfrak{o}$ auf F. Die Eigenschwingungen seien durch

$$\frac{1}{\mu}\int\limits_{G} \mathfrak{E}_n\,\overline{\mathfrak{E}}_m\,dV = \frac{1}{\varepsilon}\int\limits_{G} \mathfrak{H}_n\,\overline{\mathfrak{H}}_m\,dV = \delta_{nm}$$

normiert.

Dann gilt für $\mathfrak{x} \in G$, $r_{\mathfrak{x}} \geq \alpha > 0$ und $x \to \infty$:

$$\sum_{0 < \omega_n \leq x} |\mathfrak{E}_n(\mathfrak{x})|^2 = \frac{\mu\,(\varepsilon\,\mu)^{\frac{3}{2}}}{3\,\pi^2}\,x^3 + O\left(\frac{1}{\alpha}\,x^2\right)$$

$$\sum_{0 < \omega_n \leq x} |\mathfrak{H}_n(\mathfrak{x})|^2 = \frac{\varepsilon\,(\varepsilon\,\mu)^{\frac{3}{2}}}{3\,\pi^2}\,x^3 + O\left(\frac{1}{\alpha}\,x^2\right).$$

Die Reihen $\sum\limits_{n=1}^{\infty} \dfrac{|\mathfrak{E}_n(\mathfrak{x})|^2}{\omega_n^{2\delta}}$ *und* $\sum\limits_{n=1}^{\infty} \dfrac{|\mathfrak{H}_n(\mathfrak{x})|^2}{\omega_n^{2\delta}}$ *konvergieren für $\delta > \frac{3}{2}$ und divergieren für $\delta \leq \frac{3}{2}$.*

Ferner ist für $\mathfrak{x} \neq \mathfrak{y}$ und $x \to \infty$:

$$\sum_{0 < \omega_n \leq x} \frac{1}{\omega_n^2}\,\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}{}_n^{\sim}(\mathfrak{y}) = O(1)$$

$$\sum_{0 < \omega_n \leq x} \frac{1}{\omega_n^2}\,\mathfrak{H}_n(\mathfrak{x})\,\overline{\mathfrak{H}}{}_n^{\sim}(\mathfrak{y}) = O(1).$$

Die Reihen $\sum\limits_{n=1}^{\infty} \dfrac{1}{\omega_n^{2\delta}}\,\mathfrak{E}_n(\mathfrak{x})\,\overline{\mathfrak{E}}{}_n^{\sim}(\mathfrak{y})$ *und* $\sum\limits_{n=1}^{\infty} \dfrac{1}{\omega_n^{2\delta}}\,\mathfrak{H}_n(\mathfrak{x})\,\overline{\mathfrak{H}}{}_n^{\sim}(\mathfrak{y})$ *konvergieren für $\delta > 1$.*

Für die Anzahl der Eigenfrequenzen ω_n im Intervall $0<\omega_n\leqq x$ gilt die Abschätzung

$$\sum_{0<\omega_n\leqq x} 1 = (\varepsilon\mu)^{\frac{3}{2}} \frac{V\cdot x^3}{3\pi^2} + O(x^2 \log x).$$

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Electro-magneto-optical Effects

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1. Introduction

Electro-magneto-optical effects in stationary materials are observed when the materials are placed in strong, static electric or magnetic fields and an electromagnetic wave (light) traverses the medium. The Faraday effect is an example of the class of phenomena we have in mind. Clearly, the existence of electro-magneto-optical effects in material media is direct evidence that the equations for the electromagnetic field in a material medium, unlike the equations for it in vacuum, are non-linear since the sum of two solutions generally fails to be a solution. An analysis of electro-magneto-optical effects, though based on non-linear equations for which all but the simplest exact solutions are difficult to exhibit, may be simplified and made tractable by assuming that the dynamical part of the solution (the light wave) has such weak intensity that the vectors which describe it may be treated as infinitesimals.

Accordingly, in this paper we assume that the electric displacement and magnetic intensity fields are functions of the static electric and magnetic induction fields and linear functionals of the small dynamical electric and magnetic induction fields. This starting point is derived, in the Appendix, from the assumption that for a more general class of fields, not necessarily representable as the superposition of small dynamic fields on large static fields, the electric displacement and magnetic intensity fields are non-linear functionals of the electric and magnetic induction fields.

From our initial assumption regarding the variables which enter into the constitutive equations, we derive the limitations which are imposed on them by the assumption that the material is isotropic. The isotropic constitutive equations are then applied to the study of the manner in which a plane electromagnetic wave is propagated in an isotropic material to which static electric and magnetic induction fields are simultaneously applied.

From these results we consider in greater detail the cases when the applied electric field is zero, leading to the magneto-optical effects, and that when the applied magnetic induction field is zero, leading to the electro-optical effects. The case when both the static electric and magnetic induction fields are non-zero will be given more detailed consideration in a later paper.

2. The constitutive equations for small varying fields superposed on large constant fields

We assume that there exist in a body an electric field $\boldsymbol{E}^*$ and a magnetic nduction field $\boldsymbol{B}^*$. Each of these is assumed to be the resultant of a time-independent field and a small field which may vary with time τ. Thus,

$$\boldsymbol{E}^*(\tau) = \mathscr{E} + \varepsilon\,\boldsymbol{E}(\tau)$$
and
$$\boldsymbol{B}^*(\tau) = \mathscr{B} + \varepsilon\,\boldsymbol{B}(\tau), \tag{2.1}$$

where $\mathscr{E}$ and $\mathscr{B}$ denote the time-independent fields and $\varepsilon\,\boldsymbol{E}(\tau)$ and $\varepsilon\,\boldsymbol{B}(\tau)$ the small additional time-dependent fields. ε is a small constant. The electric displacement field $\boldsymbol{D}^*(\tau)$ and the magnetic intensity field $\boldsymbol{H}^*(\tau)$ may also be expressed in the forms

$$\boldsymbol{D}^*(\tau) = \mathscr{D} + \varepsilon\,\boldsymbol{D}(\tau)$$
and
$$\boldsymbol{H}^*(\tau) = \mathscr{H} + \varepsilon\,\boldsymbol{H}(\tau), \tag{2.2}$$

where $\mathscr{D}$ and $\mathscr{H}$ are the values of $\boldsymbol{D}^*(\tau)$ and $\boldsymbol{H}^*(\tau)$ corresponding to $\varepsilon = 0$. $\mathscr{D}$ and $\mathscr{H}$ are, of course, independent of time. For brevity we shall employ the notation

$$\boldsymbol{D}(t) = \boldsymbol{D}, \quad \boldsymbol{H}(t) = \boldsymbol{H}, \quad \boldsymbol{E}(t) = \boldsymbol{E} \quad \text{and} \quad \boldsymbol{B}(t) = \boldsymbol{B}, \tag{2.3}$$

where t is a particular instant of time.

In general, we may expect that for a given material the values of $\varepsilon\boldsymbol{D}$ and $\varepsilon\boldsymbol{H}$ will depend on $\mathscr{E}$ and $\mathscr{B}$ and on the values taken by $\varepsilon\,\boldsymbol{E}(\tau)$ and $\varepsilon\,\boldsymbol{B}(\tau)$ throughout the time interval $-\infty \leqq \tau \leqq t$. We denote the components of $\boldsymbol{D}$ and $\boldsymbol{H}$ in a rectangular Cartesian coordinate system x by D_i and H_i respectively, those of $\boldsymbol{E}(\tau)$ and $\boldsymbol{B}(\tau)$ by $E_i(\tau)$ and $B_i(\tau)$ respectively and so on. Also we use the notation

$$E_i(t) = E_i \quad \text{and} \quad B_i(t) = B_i. \tag{2.4}$$

Then, each of the vector components εD_i and εH_i may be expressed as a functional of $\varepsilon E_p(\tau)$ and $\varepsilon B_p(\tau)$ over the range $-\infty \leqq \tau \leqq t$ and an ordinary function of $\mathscr{E}_p$ and $\mathscr{B}_p$, thus

$$\varepsilon D_i = \Phi_i[\varepsilon E_p(\tau), \overset{t}{\underset{\tau=-\infty}{\,}} \varepsilon B_p(\tau); \mathscr{E}_p, \mathscr{B}_p]$$
and
$$\varepsilon H_i = \Psi_i[\varepsilon E_p(\tau), \overset{t}{\underset{\tau=-\infty}{\,}} \varepsilon B_p(\tau); \mathscr{E}_p, \mathscr{B}_p]. \tag{2.5}$$

3. Restrictions on the constitutive equations for hemihedral isotropic materials

If the material is hemihedral isotropic in the absence of any applied fields, then the constructive equations (2.5) must be form-invariant under the proper orthogonal group of transformations. Following a procedure similar to that adopted in previous papers [*1, 2, 3*], it can be shown that the restrictions imposed on Φ_i and Ψ_i by this form-invariance may be expressed in the forms

$$\Phi_i[\varepsilon E'_p(\tau), \overset{t}{\underset{\tau=-\infty}{\,}} \varepsilon B'_p(\tau); \mathscr{E}'_p, \mathscr{B}'_p] = a_{ij}\,\Phi_j[\varepsilon E_p(\tau), \overset{t}{\underset{\tau=-\infty}{\,}} \varepsilon B_p(\tau); \mathscr{E}_p, \mathscr{B}_p]$$
and
$$\Psi_i[\varepsilon E'_p(\tau), \overset{t}{\underset{\tau=-\infty}{\,}} \varepsilon B'_p(\tau); \mathscr{E}'_p, \mathscr{B}'_p] = a_{ij}\,\Psi_j[\varepsilon E_p(\tau), \overset{t}{\underset{\tau=-\infty}{\,}} \varepsilon B_p(\tau); \mathscr{E}_p, \mathscr{B}_p], \tag{3.1}$$

where

$$E'_p(\tau) = a_{pq}\, E_q(\tau), \qquad B'_p(\tau) = a_{pq}\, B_q(\tau), \\ \mathcal{E}'_p = a_{pq}\, \mathcal{E}_q \quad \text{and} \quad \mathcal{B}'_p = a_{pq}\, \mathcal{B}_q, \tag{3.2}$$

and a_{pq} are real constants satisfying the equations

$$a_{ps}\, a_{qs} = a_{sp}\, a_{sq} = \delta_{pq}, \qquad |a_{pq}| = 1. \tag{3.3}$$

We introduce the notation

$$E_p(t) = E_p = E_p^{(0)}, \qquad B_p(t) = B_p = B_p^{(0)}, \\ \frac{d^\alpha E_p(t)}{dt^\alpha} = E_p^{(\alpha)}, \qquad \frac{d^\alpha B_p(t)}{dt^\alpha} = B_p^{(\alpha)} \qquad (\alpha = 1, 2, \ldots, \mu). \tag{3.4}$$

In order to obtain the restrictions imposed on Φ_i and Ψ_i by the relations (3.1) in more explicit form, we assume that, in (2.5), Φ_i and Ψ_i are continuous hereditary functionals of $\varepsilon E_p(\tau)$ and $\varepsilon B_p(\tau)$ over the range $-\infty \leqq \tau \leqq t$ and polynomials in $\varepsilon E_p^{(\alpha)}$ and $\varepsilon B_p^{(\alpha)}$ ($\alpha = 0, 1, 2, \ldots, \mu$) and in $\mathcal{E}_p$ and $\mathcal{B}_p$. We also assume that ε is small enough so that we may neglect terms in Φ_i and Ψ_i which are nonlinear in ε. We use the notation

and

$$\boldsymbol{\Phi} = (\Phi_j), \qquad \boldsymbol{\Psi} = (\Psi_j), \qquad \boldsymbol{E}^{(\alpha)} = (E_j^{(\alpha)}) \\ \boldsymbol{B}^{(\alpha)} = (B_j^{(\alpha)}) \qquad (\alpha = 0, 1, 2, \ldots, \mu). \tag{3.5}$$

Then, following a procedure similar to that adopted in previous papers and bearing in mind that $\boldsymbol{\Phi} = \boldsymbol{\Psi} = 0$ when $\varepsilon = 0$, it can be shown that the vectors $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ and hence the vectors $\varepsilon \boldsymbol{D}$ and $\varepsilon \boldsymbol{B}$ may be expressed as linear combinations of vectors of the forms

$$\begin{gathered} \varepsilon \boldsymbol{E}^{(\alpha)}, \qquad \varepsilon \int_{-\infty}^{t} \chi_1(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau, \\ \varepsilon\, \boldsymbol{\mathcal{E}} \times \boldsymbol{B}^{(\alpha)}, \qquad \varepsilon\, \boldsymbol{\mathcal{B}} \times \boldsymbol{E}^{(\alpha)}, \\ \varepsilon\, \boldsymbol{\mathcal{E}} \times \int_{-\infty}^{t} \chi_2(t-\tau)\, \boldsymbol{B}(\tau)\, d\tau, \\ \varepsilon\, \boldsymbol{\mathcal{B}} \times \int_{-\infty}^{t} \chi_3(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau; \end{gathered} \tag{3.6}$$

vectors of the forms

$$\begin{gathered} \varepsilon \boldsymbol{B}^{(\alpha)}, \qquad \varepsilon \int_{-\infty}^{t} \chi_4(t-\tau)\, \boldsymbol{B}(\tau)\, d\tau, \\ \varepsilon\, \boldsymbol{\mathcal{E}} \times \boldsymbol{E}^{(\alpha)}, \qquad \varepsilon\, \boldsymbol{\mathcal{B}} \times \boldsymbol{B}^{(\alpha)}, \\ \varepsilon\, \boldsymbol{\mathcal{E}} \times \int_{-\infty}^{t} \chi_5(t-\tau)\, \boldsymbol{E}(\tau)\, d\tau, \\ \varepsilon\, \boldsymbol{\mathcal{B}} \times \int_{-\infty}^{t} \chi_6(t-\tau)\, \boldsymbol{B}(\tau)\, d\tau; \end{gathered} \tag{3.7}$$

$$\boldsymbol{\mathcal{E}}, \qquad \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}} \tag{3.8}$$

and

$$\boldsymbol{\mathcal{B}}, \tag{3.9}$$

where $\alpha = 0, 1, 2, \ldots, \mu$.

The coefficients of the vectors (3.6) and (3.7) are polynomials in

$$\mathcal{E}\cdot\mathcal{E}, \quad \mathcal{B}\cdot\mathcal{B} \tag{3.10}$$

and

$$\mathcal{E}\cdot\mathcal{B}. \tag{3.11}$$

The coefficients of the vectors (3.8) and (3.9) are polynomials in (3.10) and (3.11) and are linear in the quantities

$$\varepsilon\,\mathcal{E}\cdot\boldsymbol{E}^{(\alpha)}, \quad \varepsilon\,\mathcal{B}\cdot\boldsymbol{B}^{(\alpha)}, \quad \varepsilon\,[\boldsymbol{E}^{(\alpha)}, \mathcal{E}, \mathcal{B}] \tag{3.12}$$

and

$$\varepsilon\,\mathcal{E}\cdot\boldsymbol{B}^{(\alpha)}, \quad \varepsilon\,\mathcal{B}\cdot\boldsymbol{E}^{(\alpha)}, \quad \varepsilon\,[\boldsymbol{B}^{(\alpha)}, \mathcal{E}, \mathcal{B}], \tag{3.13}$$

and in expressions obtained from (3.12) and (3.13) by substituting vectors of the forms $\int_{-\infty}^{t}\chi(t-\tau)\,\boldsymbol{E}(\tau)\,d\tau$ and $\int_{-\infty}^{t}\chi(t-\tau)\,\boldsymbol{B}(\tau)$ for the vectors $\boldsymbol{E}^{(\alpha)}$ and $\boldsymbol{B}^{(\alpha)}$ respectively.

4. Restrictions on the constitutive equations for holohedral isotropic materials

We note that the vectors (3.6) and (3.8) are changed in sign by the central inversion transformation while the vectors (3.7) and (3.9) are unchanged by this transformation. Similarly, the scalars (3.11) and (3.13) are changed in sign by the central inversion transformation, while the scalars (3.10) and (3.12) are unchanged.

Since the constitutive equations for a holohedral isotropic material must be form-invariant under the central inversion transformation, in the constitutive equation for $\varepsilon\boldsymbol{D}$, each of the vectors (3.6) must have as its coefficient a polynomial in

$$\mathcal{E}\cdot\mathcal{E}, \quad \mathcal{B}\cdot\mathcal{B} \quad \text{and} \quad (\mathcal{E}\cdot\mathcal{B})^2; \tag{4.1}$$

each of the vectors (3.7) must have as its coefficient an expression of the form

$$\alpha\,\mathcal{E}\cdot\mathcal{B}, \tag{4.2}$$

where α is a polynomial in the quantities (4.1); each of the vectors (3.8) must have as its coefficient an expression of the form

$$\begin{aligned}\varepsilon\sum_{\alpha}\{A_1^{(\alpha)}\mathcal{E}\cdot\boldsymbol{E}^{(\alpha)} + A_2^{(\alpha)}\mathcal{B}\cdot\boldsymbol{B}^{(\alpha)} + A_3^{(\alpha)}[\boldsymbol{E}^{(\alpha)}, \mathcal{E}, \mathcal{B}]\} +\\ + \varepsilon\,\mathcal{E}\cdot\mathcal{B}\sum_{\alpha}\{C_1^{(\alpha)}\mathcal{E}\cdot\boldsymbol{B}^{(\alpha)} + C_2^{(\alpha)}\boldsymbol{B}\cdot\boldsymbol{E}^{(\alpha)} + C_3^{(\alpha)}[\boldsymbol{B}^{(\alpha)}, \mathcal{E}, \mathcal{B}]\},\end{aligned} \tag{4.3}$$

where the A's and C's are polynomials in $\mathcal{E}\cdot\mathcal{E}$, $\mathcal{B}\cdot\mathcal{B}$ and $(\mathcal{E}\cdot\mathcal{B})^2$, the vector (3.9) has as its coefficient an expression of the form

$$\begin{aligned}\varepsilon\,\mathcal{E}\cdot\mathcal{B}\sum_{\alpha}\{F_1^{(\alpha)}\mathcal{E}\cdot\boldsymbol{E}^{(\alpha)} + F_2^{(\alpha)}\mathcal{B}\cdot\boldsymbol{B}^{(\alpha)} + F_3^{(\alpha)}[\boldsymbol{E}^{(\alpha)}, \mathcal{E}, \mathcal{B}]\} +\\ + \varepsilon\sum_{\alpha}\{G_1^{(\alpha)}\mathcal{E}\cdot\boldsymbol{B}^{(\alpha)} + G_2^{(\alpha)}\mathcal{B}\cdot\boldsymbol{E}^{(\alpha)} + G_3^{(\alpha)}[\boldsymbol{B}^{(\alpha)}, \mathcal{E}, \mathcal{B}]\},\end{aligned} \tag{4.4}$$

where the F's and G's are polynomials in $\mathcal{E}\cdot\mathcal{E}$, $\mathcal{B}\cdot\mathcal{B}$ and $(\mathcal{E}\cdot\mathcal{B})^2$.

Again, in the constitutive equation for $\varepsilon\boldsymbol{H}$, each of the vectors (3.6) must have as its coefficient an expression of the form (4.2); each of the vectors (3.7) must have as its coefficient a polynomial in the quantities (4.1); each of the vectors (3.8) must have as its coefficient an expression of the form (4.4) and the vector (3.9) must have as its coefficient an expression of the form (4.3).

5. Constitutive equations for sinusoidal time-dependent fields

We now consider that the fields $\varepsilon\boldsymbol{E}(\tau)$ and $\varepsilon\boldsymbol{B}(\tau)$ vary sinusoidally with time with angular frequency ω, so that we may write

$$\boldsymbol{E}(\tau)=\mathscr{R}\,\overline{\boldsymbol{E}}(\boldsymbol{x})\,e^{-\iota\omega\tau} \quad\text{and}\quad \boldsymbol{B}(\tau)=\mathscr{R}\,\overline{\boldsymbol{B}}(\boldsymbol{x})\,e^{-\iota\omega\tau}, \tag{5.1}$$

where $\boldsymbol{x}$ is the position vector of a generic point of space. We shall, for the remainder of this paper, follow the common convention by omitting the symbol $\mathscr{R}$. Then, introducing (5.1) into (2.1), we obtain

$$\begin{aligned}&\boldsymbol{E}^*(\tau)=\mathscr{E}+\varepsilon\,\overline{\boldsymbol{E}}(\boldsymbol{x})\,e^{-\iota\omega\tau}\\ \text{and}\qquad &\boldsymbol{B}^*(\tau)=\mathscr{B}+\varepsilon\,\overline{\boldsymbol{B}}(\boldsymbol{x})\,e^{-\iota\omega\tau}.\end{aligned} \tag{5.2}$$

Introducing (5.1) into the constitutive equations for a holohedral isotropic material discussed in §4, we obtain

$$\boldsymbol{D}(\tau)=\overline{\boldsymbol{D}}(\boldsymbol{x})\,e^{-\iota\omega\tau} \quad\text{and}\quad \boldsymbol{H}(\tau)=\overline{\boldsymbol{H}}(\boldsymbol{x})\,e^{-\iota\omega\tau}, \tag{5.3}$$

where $\overline{\boldsymbol{D}}$ and $\overline{\boldsymbol{H}}$ are given by expressions of the form

$$\begin{aligned}\overline{\boldsymbol{D}}=\;&\alpha_1\overline{\boldsymbol{E}}+\alpha_2\mathscr{B}\times\overline{\boldsymbol{E}}+\alpha_3\mathscr{E}\times\overline{\boldsymbol{B}}+\\ &+\mathscr{E}\cdot\mathscr{B}\,(\alpha_4\overline{\boldsymbol{B}}+\alpha_5\mathscr{E}\times\overline{\boldsymbol{E}}+\alpha_6\mathscr{B}\times\overline{\boldsymbol{B}})+\\ &+\{\alpha_7\mathscr{E}\cdot\overline{\boldsymbol{E}}+\alpha_8\mathscr{B}\cdot\overline{\boldsymbol{B}}+\alpha_{19}[\overline{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}+\\ &+\{\alpha_9\mathscr{E}\cdot\overline{\boldsymbol{E}}+\alpha_{10}\mathscr{B}\cdot\overline{\boldsymbol{B}}+\alpha_{20}[\overline{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}\times\mathscr{B}+\\ &+\mathscr{E}\cdot\mathscr{B}\,\{\alpha_{11}\mathscr{E}\cdot\overline{\boldsymbol{B}}+\alpha_{12}\mathscr{B}\cdot\overline{\boldsymbol{E}}+\alpha_{21}[\overline{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}+\\ &+\mathscr{E}\cdot\mathscr{B}\,\{\alpha_{13}\mathscr{E}\cdot\overline{\boldsymbol{B}}+\alpha_{14}\mathscr{B}\cdot\overline{\boldsymbol{E}}+\alpha_{22}[\overline{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}\times\mathscr{B}+\\ &+\{\alpha_{15}\mathscr{E}\cdot\overline{\boldsymbol{B}}+\alpha_{16}\mathscr{B}\cdot\overline{\boldsymbol{E}}+\alpha_{23}[\overline{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\}\,\mathscr{B}+\\ &+\mathscr{E}\cdot\mathscr{B}\,\{\alpha_{17}\mathscr{E}\cdot\overline{\boldsymbol{E}}+\alpha_{18}\mathscr{B}\cdot\overline{\boldsymbol{B}}+\alpha_{24}[\overline{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\}\,\mathscr{B}\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\overline{\boldsymbol{H}}=\;&\beta_1\overline{\boldsymbol{B}}+\beta_2\mathscr{E}\times\overline{\boldsymbol{E}}+\beta_3\mathscr{B}\times\overline{\boldsymbol{B}}+\\ &+\mathscr{E}\cdot\mathscr{B}\,(\beta_4\overline{\boldsymbol{E}}+\beta_5\mathscr{B}\times\overline{\boldsymbol{E}}+\beta_6\mathscr{E}\times\overline{\boldsymbol{B}})+\\ &+\{\beta_7\mathscr{E}\cdot\overline{\boldsymbol{B}}+\beta_8\mathscr{B}\cdot\overline{\boldsymbol{E}}+\beta_{19}[\overline{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}+\\ &+\{\beta_9\mathscr{E}\cdot\overline{\boldsymbol{B}}+\beta_{10}\mathscr{B}\cdot\overline{\boldsymbol{E}}+\beta_{20}[\overline{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}\times\mathscr{B}+\\ &+\mathscr{E}\cdot\mathscr{B}\,\{\beta_{11}\mathscr{E}\cdot\overline{\boldsymbol{E}}+\beta_{12}\mathscr{B}\cdot\overline{\boldsymbol{B}}+\beta_{21}[\overline{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}+\\ &+\mathscr{E}\cdot\mathscr{B}\,\{\beta_{13}\mathscr{E}\cdot\overline{\boldsymbol{E}}+\beta_{14}\mathscr{B}\cdot\overline{\boldsymbol{B}}+\beta_{22}[\overline{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\}\,\mathscr{E}\times\mathscr{B}+\\ &+\{\beta_{15}\mathscr{E}\cdot\overline{\boldsymbol{E}}+\beta_{16}\mathscr{B}\cdot\overline{\boldsymbol{B}}+\beta_{23}[\overline{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\}\,\mathscr{B}+\\ &+\mathscr{E}\cdot\mathscr{B}\,\{\beta_{17}\mathscr{E}\cdot\overline{\boldsymbol{B}}+\beta_{18}\mathscr{B}\cdot\overline{\boldsymbol{E}}+\beta_{24}[\overline{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\}\,\mathscr{B},\end{aligned} \tag{5.5}$$

where the α's and β's are functions of $\iota\omega$ and polynomials in $\mathscr{E}\cdot\mathscr{E}$, $\mathscr{B}\cdot\mathscr{B}$ and $(\mathscr{E}\cdot\mathscr{B})^2$.

The following identities hold:

$$\begin{aligned}
[\bar{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\,\mathscr{E} &\equiv -(\mathscr{E}\cdot\mathscr{B})\,\mathscr{E}\times\bar{\boldsymbol{E}}+(\mathscr{E}\cdot\mathscr{E})\,\mathscr{B}\times\bar{\boldsymbol{E}}+(\mathscr{E}\cdot\bar{\boldsymbol{E}})\,\mathscr{E}\times\mathscr{B},\\
[\bar{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\,\mathscr{E} &\equiv -(\mathscr{E}\cdot\mathscr{B})\,\mathscr{E}\times\bar{\boldsymbol{B}}-(\mathscr{E}\cdot\mathscr{E})\,\mathscr{B}\times\bar{\boldsymbol{B}}+(\mathscr{E}\cdot\bar{\boldsymbol{B}})\,\mathscr{E}\times\mathscr{B},\\
[\bar{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\,\mathscr{B} &\equiv (\mathscr{E}\cdot\mathscr{B})\,\mathscr{B}\times\bar{\boldsymbol{E}}-(\mathscr{B}\cdot\mathscr{B})\,\mathscr{E}\times\bar{\boldsymbol{E}}-(\mathscr{B}\cdot\bar{\boldsymbol{E}})\,\mathscr{E}\times\mathscr{B},\\
[\bar{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\,\mathscr{B} &\equiv (\mathscr{E}\cdot\mathscr{B})\,\mathscr{B}\times\bar{\boldsymbol{B}}+(\mathscr{B}\cdot\mathscr{B})\,\mathscr{E}\times\bar{\boldsymbol{B}}-(\mathscr{B}\cdot\bar{\boldsymbol{B}})\,\mathscr{E}\times\mathscr{B}.
\end{aligned}\tag{5.6}$$

From the third of these we obtain

$$\begin{aligned}
[\bar{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\,\mathscr{E}\times\mathscr{B} \equiv (\mathscr{E}\cdot\mathscr{B})\,\mathscr{E}\times(\mathscr{B}\times\bar{\boldsymbol{E}})-(\mathscr{B}\cdot\mathscr{B})\,\mathscr{E}\times(\mathscr{E}\times\bar{\boldsymbol{E}})\,-\\
-(\mathscr{B}\cdot\bar{\boldsymbol{E}})\,\mathscr{E}\times(\mathscr{E}\times\mathscr{B}).
\end{aligned}\tag{5.7}$$

Using identities of the type

$$\mathscr{E}\times(\mathscr{B}\times\bar{\boldsymbol{E}})\equiv(\mathscr{E}\cdot\bar{\boldsymbol{E}})\,\mathscr{B}-(\mathscr{E}\cdot\mathscr{B})\,\bar{\boldsymbol{E}},\tag{5.8}$$

we obtain from (5.7),

$$\begin{aligned}
[\bar{\boldsymbol{E}},\mathscr{E},\mathscr{B}]\,\mathscr{E}\times\mathscr{B} \equiv (\mathscr{E}\cdot\mathscr{B})\,[(\mathscr{E}\cdot\bar{\boldsymbol{E}})\,\mathscr{B}-(\mathscr{E}\cdot\mathscr{B})\,\bar{\boldsymbol{E}}]\,-\\
-(\mathscr{B}\cdot\mathscr{B})\,[(\mathscr{E}\cdot\bar{\boldsymbol{E}})\,\mathscr{E}-(\mathscr{E}\cdot\mathscr{E})\,\bar{\boldsymbol{E}}]\,-\\
-(\mathscr{B}\cdot\bar{\boldsymbol{E}})\,[(\mathscr{E}\cdot\mathscr{B})\,\mathscr{E}-(\mathscr{E}\cdot\mathscr{E})\,\mathscr{B}].
\end{aligned}\tag{5.9}$$

Similarly, we have

$$\begin{aligned}
[\bar{\boldsymbol{B}},\mathscr{E},\mathscr{B}]\,\mathscr{E}\times\mathscr{B} \equiv (\mathscr{E}\cdot\mathscr{B})\,[(\mathscr{E}\cdot\bar{\boldsymbol{B}})\,\mathscr{B}-(\mathscr{E}\cdot\mathscr{B})\,\bar{\boldsymbol{B}}]\,-\\
-(\mathscr{B}\cdot\mathscr{B})\,[(\mathscr{E}\cdot\bar{\boldsymbol{B}})\,\mathscr{E}-(\mathscr{E}\cdot\mathscr{E})\,\bar{\boldsymbol{B}}]\,-\\
-(\mathscr{B}\cdot\bar{\boldsymbol{B}})\,[(\mathscr{E}\cdot\mathscr{B})\,\mathscr{E}-(\mathscr{E}\cdot\mathscr{E})\,\mathscr{B}].
\end{aligned}\tag{5.10}$$

Using the relations (5.6), (5.9) and (5.10), it is clear that we may, without loss of generality, take $\alpha_{19}=\alpha_{20}=\cdots=\alpha_{24}=0$ and $\beta_{19}=\beta_{20}=\cdots=\beta_{24}=0$ in equations (5.4) and (5.5).

6. Constitutive equations for a plane wave

We now consider that the time-dependent fields are those for an infinite plane wave, with angular frequency ω and wave number k, in the direction of the unit vector $\boldsymbol{n}$, and are given by

$$\begin{aligned}
\boldsymbol{E}&=\boldsymbol{e}\,e^{\iota(k\boldsymbol{n}\cdot\boldsymbol{x}-\omega t)},\\
\boldsymbol{B}&=\boldsymbol{b}\,e^{\iota(k\boldsymbol{n}\cdot\boldsymbol{x}-\omega t)},\\
\boldsymbol{D}&=\boldsymbol{d}\,e^{\iota(k\boldsymbol{n}\cdot\boldsymbol{x}-\omega t)},\\
\boldsymbol{H}&=\boldsymbol{h}\,e^{\iota(k\boldsymbol{n}\cdot\boldsymbol{x}-\omega t)},
\end{aligned}\tag{6.1}$$

where the complex vectors $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{h}$ are independent of position and time. $\boldsymbol{x}$ denotes the position vector of a generic point of space.

In this case, we may replace $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{B}}$, $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{H}}$ by $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{h}$ in the constitutive equations (6.1), in which we take $\alpha_{19}=\alpha_{20}=\cdots=\alpha_{24}=0$ and $\beta_{19}=\beta_{20}=\cdots=\beta_{24}=0$. These equations may then be re-written as

$$\begin{aligned}
\boldsymbol{d}&=\boldsymbol{\Phi}\cdot\boldsymbol{e}+\boldsymbol{\Psi}\cdot\boldsymbol{b},\\
\boldsymbol{h}&=\boldsymbol{\Omega}\cdot\boldsymbol{e}+\boldsymbol{\Lambda}\cdot\boldsymbol{b},
\end{aligned}\tag{6.2}$$

where $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ are the 3×3 matrices defined by $\mathbf{\Phi}=\|\Phi_{ij}\|$, $\mathbf{\Psi}=\|\Psi_{ij}\|$, … and

$$
\begin{aligned}
\Phi_{ij} = {} & \alpha_1\,\delta_{ij} - \alpha_2\,\varepsilon_{ijk}\,\mathcal{B}_k - \alpha_5\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ijk}\,\mathcal{E}_k + \alpha_7\,\mathcal{E}_i\,\mathcal{E}_j + \\
& + \alpha_9\,\varepsilon_{imn}\,\mathcal{E}_m\,\mathcal{B}_n\,\mathcal{E}_j + \alpha_{12}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{E}_i\,\mathcal{B}_j + \\
& + \alpha_{14}\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ipq}\,\mathcal{E}_p\,\mathcal{B}_q\,\mathcal{B}_j + \alpha_{16}\,\mathcal{B}_i\,\mathcal{B}_j + \\
& + \alpha_{17}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{B}_i\,\mathcal{E}_j, \\[1ex]
\Psi_{ij} = {} & -\alpha_3\,\varepsilon_{ijk}\,\mathcal{E}_k + \alpha_4\,\mathcal{E}_m\,\mathcal{B}_m\,\delta_{ij} - \alpha_6\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ijk}\,\mathcal{B}_k + \\
& + \alpha_8\,\mathcal{E}_i\,\mathcal{B}_j + \alpha_{10}\,\varepsilon_{imn}\,\mathcal{E}_m\,\mathcal{B}_n\,\mathcal{B}_j + \\
& + \alpha_{11}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{E}_i\,\mathcal{E}_j + \alpha_{13}\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ipq}\,\mathcal{E}_p\,\mathcal{B}_q\,\mathcal{E}_j + \\
& + \alpha_{15}\,\mathcal{B}_i\,\mathcal{E}_j + \alpha_{18}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{B}_i\,\mathcal{B}_j, \qquad (6.3) \\[1ex]
\Omega_{ij} = {} & -\beta_2\,\varepsilon_{ijk}\,\mathcal{E}_k + \beta_4\,\mathcal{E}_m\,\mathcal{B}_m\,\delta_{ij} - \beta_5\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ijk}\,\mathcal{B}_k + \\
& + \beta_8\,\mathcal{E}_i\,\mathcal{B}_j + \beta_{10}\,\varepsilon_{imn}\,\mathcal{E}_m\,\mathcal{B}_n\,\mathcal{B}_j + \beta_{11}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{E}_i\,\mathcal{E}_j + \\
& + \beta_{13}\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ipq}\,\mathcal{E}_p\,\mathcal{B}_q\,\mathcal{E}_i + \beta_{15}\,\mathcal{B}_i\,\mathcal{E}_j + \\
& + \beta_{18}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{B}_i\,\mathcal{B}_j, \\[1ex]
\Lambda_{ij} = {} & \beta_1\,\delta_{ij} - \beta_3\,\varepsilon_{ijk}\,\mathcal{B}_k - \beta_6\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ijk}\,\mathcal{E}_k + \beta_7\,\mathcal{E}_i\,\mathcal{E}_j + \\
& + \beta_9\,\varepsilon_{imn}\,\mathcal{E}_m\,\mathcal{B}_n\,\mathcal{E}_j + \beta_{12}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{E}_i\,\mathcal{B}_j + \\
& + \beta_{14}\,\mathcal{E}_m\,\mathcal{B}_m\,\varepsilon_{ipq}\,\mathcal{E}_p\,\mathcal{B}_q\,\mathcal{B}_j + \beta_{16}\,\mathcal{B}_i\,\mathcal{B}_j + \beta_{17}\,\mathcal{E}_m\,\mathcal{B}_m\,\mathcal{B}_i\,\mathcal{E}_j .
\end{aligned}
$$

δ_{ij} denotes the Kronecker delta and ε_{ijk} the alternating symbol, and the α's and β's are functions of $\iota\omega$ and polynomials in $\mathcal{E}\cdot\mathcal{E}$, $\mathcal{B}\cdot\mathcal{B}$ and $(\mathcal{E}\cdot\mathcal{B})^2$.

7. The propagation of a plane electromagnetic wave

The electromagnetic field equations in the absence of free currents and charges are

$$
\begin{aligned}
&\frac{\partial \boldsymbol{B}^*}{\partial t} + \operatorname{curl}\boldsymbol{E}^* = 0, \qquad \operatorname{div}\boldsymbol{B}^* = 0, \\
&\operatorname{curl}\boldsymbol{H}^* - \frac{\partial \boldsymbol{D}^*}{\partial t} = 0, \qquad \operatorname{div}\boldsymbol{D}^* = 0.
\end{aligned}
\tag{7.1}
$$

We now assume that $\boldsymbol{E}^*$ and $\boldsymbol{B}^*$ are given by (2.1) and that the static portions of these fields, $\mathcal{E}$ and $\mathcal{B}$, are independent of position. We then see that $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$ must satisfy the equations

$$
\begin{aligned}
&\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl}\boldsymbol{E} = 0, \qquad \operatorname{div}\boldsymbol{B} = 0, \\
&\operatorname{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = 0, \qquad \operatorname{div}\boldsymbol{D} = 0.
\end{aligned}
\tag{7.2}
$$

For the infinite plane wave (6.1), we obtain

$$
k\boldsymbol{n}\times\boldsymbol{e} - \omega\,\boldsymbol{b} = 0 \quad \text{and} \quad k\boldsymbol{n}\times\boldsymbol{h} + \omega\,\boldsymbol{d} = 0. \tag{7.3}
$$

Introducing (6.2) into the second of equations (7.3), we obtain, with the notation* $\eta = k/\omega$,

$$\eta \boldsymbol{n} \times (\boldsymbol{\Omega} \cdot \boldsymbol{e} + \boldsymbol{\Lambda} \cdot \boldsymbol{b}) + \boldsymbol{\Phi} \cdot \boldsymbol{e} + \boldsymbol{\Psi} \cdot \boldsymbol{b} = 0. \tag{7.4}$$

Using the first of equations (7.3) to substitute for $\boldsymbol{b}$ in (7.4), we obtain

$$\eta \boldsymbol{n} \times [\boldsymbol{\Omega} \cdot \boldsymbol{e} + \eta \boldsymbol{\Lambda} \cdot (\boldsymbol{n} \times \boldsymbol{e})] + \eta \boldsymbol{\Psi} \cdot (\boldsymbol{n} \times \boldsymbol{e}) + \boldsymbol{\Phi} \cdot \boldsymbol{e} = 0. \tag{7.5}$$

(7.5) is a system of three homogeneous linear equations in the components of $\boldsymbol{e}$. We may therefore write it as

$$\chi_{ij} e_j = 0, \tag{7.6}$$

where

$$\chi_{ij} = \Phi_{ij} + \eta (\varepsilon_{jkl} \Psi_{ik} n_l - \varepsilon_{ikl} \Omega_{kj} n_l) + \eta^2 \varepsilon_{jkl} \varepsilon_{ipq} \Lambda_{qk} n_p n_l. \tag{7.7}$$

The secular equation for η is

$$|\chi_{ij}| = 0. \tag{7.8}$$

If the wave is propagated in the direction of the x_3-axis of a rectangular Cartesian coordinate system x, we may take $n_i = \delta_{3i}$ in (7.7). We then obtain from (7.8),

$$\begin{vmatrix} \Phi_{11} + \eta (\Psi_{12} - \Omega_{21}) - \eta^2 \Lambda_{22}, & \Phi_{12} - \eta (\Psi_{11} + \Omega_{22}) + \eta^2 \Lambda_{21}, & \Phi_{13} - \eta^2 \Omega_{23} \\ \Phi_{21} + \eta (\Psi_{22} + \Omega_{11}) + \eta^2 \Lambda_{12}, & \Phi_{22} + \eta (\Psi_{21} + \Omega_{12}) - \eta^2 \Lambda_{11}, & \Phi_{23} + \eta \Omega_{13} \\ \Phi_{31} + \eta \Psi_{32}, & \Phi_{32} - \eta \Psi_{31}, & \Phi_{33} \end{vmatrix} = 0. \tag{7.9}$$

This is the secular equation for an infinite plane electromagnetic wave propagated in a holohedral isotropic material to which static electric and magnetic induction fields are applied. We note that it is an equation of fourth degree in η. In general the solutions of this equation are complex.

Denoting η by

$$\eta = \eta^+ + \iota \eta^-, \tag{7.10}$$

where η^+ and η^- are real, we see that a solution with $\eta^+ > 0$ corresponds to a wave travelling in the positive direction of the unit vector $\boldsymbol{n}$, while a solution with $\eta^+ < 0$ corresponds to a wave travelling in the negative direction of $\boldsymbol{n}$. The amplitude of the wave will increase in the direction of propagation of the wave for the first of these waves if $\eta^- < 0$ and for the second of the waves if $\eta^- > 0$. Consequently, if either of these situations arises for any value of ω, the system will be unstable. We shall in our subsequent discussion assume that such situations do not arise.

Thus, the four solutions of (7.9) correspond, in general, to four elliptically-polarized, skew $\boldsymbol{E}$-waves travelling with different complex velocities. The manner in which the plane of polarization and ellipticity can be calculated is shown in the next section.

8. Analysis of a skew, elliptically-polarized plane wave

We consider a plane wave given by

$$\boldsymbol{E} = \boldsymbol{e} e^{\iota \varphi}, \tag{8.1}$$

where

$$\varphi = k \boldsymbol{n} \cdot \boldsymbol{x} - \omega t. \tag{8.2}$$

---

* $1/\eta$ is then the complex velocity.

We have*
$$\boldsymbol{E}^+ = \boldsymbol{e}^+ \cos\varphi - \boldsymbol{e}^- \sin\varphi. \tag{8.3}$$

The vector $\boldsymbol{E}^+$ therefore executes an ellipse. Let $\boldsymbol{l}$ be a unit vector which is at all times perpendicular to the plane of the ellipse. $\boldsymbol{l}$ is taken in the same sense in relation to the plane of polarization as $\boldsymbol{n}$, *i.e.* $\boldsymbol{l}\cdot\boldsymbol{n} \geqq 0$. Then, $\boldsymbol{l}\cdot\boldsymbol{E}^+ = 0$. From (8.3), we then have
$$\boldsymbol{l}\cdot\boldsymbol{e}^+ = \boldsymbol{l}\cdot\boldsymbol{e}^- = 0, \tag{8.4}$$
whence
$$\boldsymbol{l} = \pm \frac{\boldsymbol{e}^+ \times \boldsymbol{e}^-}{[(\boldsymbol{e}^+ \times \boldsymbol{e}^-)\cdot(\boldsymbol{e}^+ \times \boldsymbol{e}^-)]^{\frac{1}{2}}} = \pm \frac{\boldsymbol{e} \times \boldsymbol{e}^*}{[(\boldsymbol{e} \times \boldsymbol{e}^*)\cdot(\boldsymbol{e} \times \boldsymbol{e}^*)]^{\frac{1}{2}}}, \tag{8.5}$$
the positive or negative signs being chosen to make $\boldsymbol{l}\cdot\boldsymbol{n} \geqq 0$. From (8.3), we have
$$\boldsymbol{E}^+\cdot\boldsymbol{E}^+ = \boldsymbol{e}^+\cdot\boldsymbol{e}^+ \cos^2\varphi - 2\boldsymbol{e}^+\cdot\boldsymbol{e}^- \cos\varphi \sin\varphi + \boldsymbol{e}^-\cdot\boldsymbol{e}^- \sin^2\varphi. \tag{8.6}$$

We note that $\boldsymbol{E}^+\cdot\boldsymbol{E}^+$ is the square of the radius vector to a point on the ellipse. The vector $\boldsymbol{E}^+$ is in the direction of an axis of the ellipse when $d(\boldsymbol{E}^+\cdot\boldsymbol{E}^+)/d\varphi = 0$, *i.e.* when
$$\tan 2\varphi = -\frac{2\boldsymbol{e}^+\cdot\boldsymbol{e}^-}{\boldsymbol{e}^+\cdot\boldsymbol{e}^+ - \boldsymbol{e}^-\cdot\boldsymbol{e}^-} = \frac{\iota(\boldsymbol{e}\cdot\boldsymbol{e} - \boldsymbol{e}^*\cdot\boldsymbol{e}^*)}{\boldsymbol{e}\cdot\boldsymbol{e} + \boldsymbol{e}^*\cdot\boldsymbol{e}^*}. \tag{8.7}$$

From (8.6) and (8.7), we see that the semi-axes L_1 and L_2 of the ellipse are given by
$$\begin{aligned} L_1^2, L_2^2 &= \tfrac{1}{2}(\boldsymbol{e}^+\cdot\boldsymbol{e}^+ + \boldsymbol{e}^-\cdot\boldsymbol{e}^-) \pm \tfrac{1}{2}\{(\boldsymbol{e}^+\cdot\boldsymbol{e}^+ - \boldsymbol{e}^-\cdot\boldsymbol{e}^-)^2 + 4(\boldsymbol{e}^+\cdot\boldsymbol{e}^-)^2\}^{\frac{1}{2}} \\ &= \tfrac{1}{2}\boldsymbol{e}\cdot\boldsymbol{e}^* \pm \tfrac{1}{2}\{(\boldsymbol{e}\cdot\boldsymbol{e})(\boldsymbol{e}^*\cdot\boldsymbol{e}^*)\}^{\frac{1}{2}}. \end{aligned} \tag{8.8}$$

$\boldsymbol{l}$ and $\boldsymbol{e}$ may satisfy certain special relations which correspond to the $\boldsymbol{E}^+$-wave having a particular character. These relations together with the nature of the wave to which they correspond are listed below:

$$[\boldsymbol{e}^+, \boldsymbol{e}^-, \boldsymbol{l}] > 0 \quad \text{right-handed wave,} \tag{8.9}$$

$$[\boldsymbol{e}^+, \boldsymbol{e}^-, \boldsymbol{l}] < 0 \quad \text{left-handed wave,} \tag{8.10}$$

$$\boldsymbol{e}\cdot\boldsymbol{e} = 0 \quad \text{circularly-polarized wave,} \tag{8.11}$$

$$\boldsymbol{e}^* \times \boldsymbol{e} = 0 \quad \text{linearly-polarized wave,} \tag{8.12}$$

$$\boldsymbol{l} = \boldsymbol{n}, \quad \text{and hence} \quad \boldsymbol{e}\cdot\boldsymbol{n} = 0, \quad \text{transverse wave,} \tag{8.13}$$

$$\boldsymbol{e} \times \boldsymbol{n} = 0 \quad \text{longitudinal wave.} \tag{8.14}$$

9. Magneto-optical effects in holohedral isotropic materials

In this section, we shall discuss the interaction between a plane electromagnetic wave in a holohedral isotropic material and a static magnetic field applied to the material. We accordingly take $\mathscr{E} = 0$. From equations (6.3), we then obtain
$$\begin{aligned} \Phi_{ij} &= \alpha_1 \delta_{ij} - \alpha_2 \varepsilon_{ijk} \mathscr{B}_k + \alpha_{16} \mathscr{B}_i \mathscr{B}_j, \\ \Lambda_{ij} &= \beta_1 \delta_{ij} - \beta_3 \varepsilon_{ijk} \mathscr{B}_k + \beta_{16} \mathscr{B}_i \mathscr{B}_j \end{aligned} \tag{9.1}$$

---

* Here and subsequently, we shall denote the real and imaginary parts of a complex quantity by using the superscripts + and −.

and
$$\Psi_{ij}=\Omega_{ij}=0,$$
the α's and β's being functions of $\iota\omega$ and polynomials in $\mathscr{B}_m\,\mathscr{B}_m$.

From (7.7) and (9.1), we obtain
$$\chi_{ij}=\Phi_{ij}+\eta^2\,\varepsilon_{jkl}\,\varepsilon_{ipq}\,\Lambda_{qk}\,n_p\,n_l. \tag{9.2}$$

a) Longitudinal magnetic field

If the magnetic induction field $\mathscr{B}$ is parallel to the direction of propagation of the wave, then we may without loss of generality take both of these directions as the 3-direction of a rectangular Cartesian system x. We then have
$$n_i=\delta_{3i} \quad \text{and} \quad \mathscr{B}_i=\delta_{3i}. \tag{9.3}$$
With (9.3), the expressions (9.1) for Φ_{ij} and Λ_{ij} become
$$\begin{aligned}\Phi_{ij}&=\alpha_1\,\delta_{ij}-\alpha_2\,\mathscr{B}\,\varepsilon_{ij3}+\alpha_{16}\,\mathscr{B}^2\,\delta_{3i}\,\delta_{3j}\\ \text{and}\qquad \Lambda_{ij}&=\beta_1\,\delta_{ij}-\beta_3\,\mathscr{B}\,\varepsilon_{ij3}+\beta_{16}\,\mathscr{B}^2\,\delta_{3i}\,\delta_{3j},\end{aligned} \tag{9.4}$$
where the α's and β's are polynomials in $\mathscr{B}^2$. From (9.2) and (9.4), we obtain
$$\chi_{ij}=\alpha_1\,\delta_{ij}-\alpha_2\,\mathscr{B}\,\varepsilon_{ij3}+\alpha_{16}\,\mathscr{B}^2\,\delta_{3i}\,\delta_{3j}+\eta^2\left[-\beta_1\,(\delta_{ij}-\delta_{3i}\,\delta_{3j})+\beta_3\,\mathscr{B}\,\varepsilon_{ij3}\right]. \tag{9.5}$$
Whence,
$$\|\chi_{ij}\|=\begin{Vmatrix}\alpha_1-\beta_1\,\eta^2, & -(\alpha_2-\beta_3\,\eta^2)\,\mathscr{B}, & 0\\ (\alpha_2-\beta_3\,\eta^2)\,\mathscr{B}, & \alpha_1-\beta_1\,\eta^2, & 0\\ 0, & 0, & \alpha_1+\alpha_{16}\,\mathscr{B}^2\end{Vmatrix}. \tag{9.6}$$
The secular equation (7.8) then yields
$$\begin{aligned}&\alpha_1+\alpha_{16}\,\mathscr{B}^2=0\\ \text{or}\qquad &(\alpha_1-\beta_1\,\eta^2)^2+(\alpha_2-\beta_3\,\eta^2)^2\,\mathscr{B}^2=0.\end{aligned} \tag{9.7}$$
Discounting the first alternative, we obtain
$$\eta^2=\frac{\alpha_1+i\,\alpha_2\,\mathscr{B}}{\beta_1+i\,\beta_3\,\mathscr{B}} \quad \text{or} \quad \frac{\alpha_1-i\,\alpha_2\,\mathscr{B}}{\beta_1-i\,\beta_3\,\mathscr{B}}. \tag{9.8}$$

From (9.6) and (7.6), it is easily seen that $e_3=0$. Consequently, since $n_i=\delta_{3i}$, the relation (8.13) is satisfied. Also, unless $(\alpha_2-\beta_3\eta^2)\mathscr{B}=0$ or $\alpha_1-\beta_1\eta^2=0$, it is seen that the relation (8.11) is satisfied. Thus each of the four solutions for η, given by (9.8), corresponds to a transverse, circularly-polarized plane wave. Two of these travel with speeds, which are in general different, in the positive direction of the x_3-axis and the remaining two travel with these speeds in the negative direction. If $\mathscr{B}>0$, the wave in the positive direction of the x_3-axis given by the first of the solutions (9.8) is a left-handed wave and that given by the second of the solutions (9.8) is a right-handed wave; on the other hand the waves in the negative direction of the x_3-axis given by the first and second of the solutions (9.8) are right-handed and left-handed respectively.

It is thus seen that the right and left circularly polarized $\boldsymbol{E}^+$ waves, with a given direction of propagation parallel to the magnetic induction field $\mathscr{B}$, travel with different speeds. This is equivalent to showing that the plane of polarization of a linearly polarized wave rotates at a rate proportional to the distance travelled. This is the well-known Faraday effect. In (9.8), α_1, β_1, α_2 and β_3 are, in general, polynomials in $\mathscr{B}^2$.

b) Transverse magnetic field

We now consider that the magnetic induction field $\boldsymbol{\mathscr{B}}$ is perpendicular to the direction of propagation $\boldsymbol{n}$ of the plane electromagnetic wave. We choose a rectangular Cartesian reference system with its 1-axis parallel to $\boldsymbol{\mathscr{B}}$ and its 3-axis parallel to $\boldsymbol{n}$. Then, in (9.1) and (9.2), we take

$$\mathscr{B}_i = \mathscr{B}\,\delta_{1i} \quad \text{and} \quad n_i = \delta_{3i}. \tag{9.9}$$

We obtain

$$\begin{aligned} \Phi_{ij} &= \alpha_1\,\delta_{ij} - \alpha_2\,\mathscr{B}\,\varepsilon_{ij1} + \alpha_{16}\,\mathscr{B}^2\,\delta_{1i}\,\delta_{1j}, \\ \Lambda_{ij} &= \beta_1\,\delta_{ij} - \beta_3\,\mathscr{B}\,\varepsilon_{ij1} + \beta_{16}\,\mathscr{B}^2\,\delta_{1i}\,\delta_{1j}, \end{aligned} \tag{9.10}$$

and

$$\begin{aligned} \chi_{ij} = {} & \alpha_1\,\delta_{ij} - \alpha_2\,\mathscr{B}\,\varepsilon_{ij1} + \alpha_{16}\,\mathscr{B}^2\,\delta_{1i}\,\delta_{1j} + \\ & + \eta^2[-\beta_1(\delta_{ij} - \delta_{3i}\,\delta_{3j}) - \beta_{16}\,\mathscr{B}^2\,\varepsilon_{i13}\,\varepsilon_{i13}]. \end{aligned} \tag{9.11}$$

Whence,

$$\|\chi_{ij}\| = \left\| \begin{matrix} \alpha_1 + \alpha_{16}\,\mathscr{B}^2 - \beta_1\,\eta^2, & 0, & 0 \\ 0, & \alpha_1 - \eta^2(\beta_1 + \beta_{16}\,\mathscr{B}^2), & -\alpha_2\,\mathscr{B} \\ 0, & \alpha_2\,\mathscr{B}, & \alpha_1 \end{matrix} \right\|. \tag{9.12}$$

The secular equation (7.8) then yields

$$\eta^2 = \frac{\alpha_1 + \alpha_{16}\,\mathscr{B}^2}{\beta_1} \tag{9.13}$$

and

$$\eta^2 = \frac{\alpha_1^2 + \alpha_2^2\,\mathscr{B}^2}{\alpha_1(\beta_1 + \beta_{16}\,\mathscr{B}^2)}. \tag{9.14}$$

If we take the solution (9.13) for η^2, we see from (9.11) and (7.6) that, in general, $e_2 = e_3 = 0$. Thus the two waves corresponding to this solution satisfy the conditions (8.12) and (8.13) and are therefore transverse waves, linearly-polarized in the x_1-direction, travelling with equal speeds in the positive and negative directions of the x_3-axis.

The solution (9.14) for η^2 yields, with (9.11) and (7.6), $e_1 = 0$. It therefore corresponds, in general, to two elliptically-polarized skew waves, the planes of polarization of which are normal to $\boldsymbol{\mathscr{B}}$, travelling with equal and opposite velocities. For these waves,

$$e_3/e_2 = -\alpha_2\,\mathscr{B}/\alpha_1, \tag{9.15}$$

so that $\boldsymbol{e}$ has the form

$$\boldsymbol{e} = \xi(0, 1, \zeta), \tag{9.16}$$

where

$$\zeta = -\alpha_2\,\mathscr{B}/\alpha_1 \tag{9.17}$$

and ξ is an arbitrary complex number. The $\boldsymbol{E}^+$-waves whose velocities are determined by (9.14) are linearly-polarized if and only if $\boldsymbol{e}^* \times \boldsymbol{e} = 0$ (*cf.* equation (8.12)). This condition is satisfied if and only if ζ is real. The waves are transverse if and only if $l_3 = 0$ (*cf.* equation (8.13)), *i.e.* if and only if $\zeta = 0$. They are circularly-polarized if and only if $\boldsymbol{e} \cdot \boldsymbol{e} = 0$ (*cf.* equation (8.11)), *i.e.* if and only if $\zeta = \pm\iota$. For all other values of ζ corresponding to the solution (9.14) for η^2, the $\boldsymbol{E}^{+}$-wave is an elliptically-polarized skew wave. The locus of $\boldsymbol{E}^{+}$ at a fixed point in the material is an ellipse in the plane normal to $\mathscr{B}$. From (8.8) the semi-major and semi-minor axes L_1 and L_2 of this ellipse are given by

$$L_1^2, L_2^2 = \tfrac{1}{2}\xi\,\xi^*\{1+\zeta\,\zeta^* \pm [(1+\zeta^2)\,(1+\zeta^{*2})]^{\frac{1}{2}}\}, \tag{9.18}$$

the positive sign applying to the semi-major axis L_1 and the negative sign to the semi-minor axis L_2.

Introducing (9.16) into equation (8.7), we obtain

$$\tan 2\varphi = \frac{\iota\,[\xi^2(1+\zeta^2)-\xi^{*2}(1+\zeta^{*2})]}{\xi^2(1+\zeta^2)+\xi^{*2}(1+\zeta^{*2})}\,. \tag{9.19}$$

Voigt [5] explained the double refraction of light propagating normal to a magnetic field by the difference between the speeds as determined by (9.13) and (9.14) and observed the effect in vapours. Cotton and Mouton (see, for example [6]) observed a very strong effect of this nature in certain liquids. The skewness and ellipticity of the $\boldsymbol{E}^{+}$-wave propagated with speed given by (9.14) seems not to have been noticed.

10. Electro-optical effects in holohedral isotropic materials

We now discuss the interaction between a plane electro-magnetic wave and a static electric field in the material through which the wave is propagated. We accordingly take $\mathscr{B}=0$. From (6.3), we then obtain

$$\begin{aligned}
\Phi_{ij} &= \alpha_1\,\delta_{ij} + \alpha_7\,\mathscr{E}_i\,\mathscr{E}_j,\\
\Psi_{ij} &= -\,\alpha_3\,\varepsilon_{ijk}\,\mathscr{E}_k,\\
\Omega_{ij} &= -\,\beta_2\,\varepsilon_{ijk}\,\mathscr{E}_k,\\
\Lambda_{ij} &= \beta_1\,\delta_{ij} + \beta_7\,\mathscr{E}_i\,\mathscr{E}_j,
\end{aligned} \tag{10.1}$$

the α's and β's being functions of $\iota\omega$ and polynomials in $\mathscr{E}_m\,\mathscr{E}_m$.

From (7.7) and (10.1), we obtain

$$\begin{aligned}
\chi_{ij} = {}&(\alpha_1\,\delta_{ij} + \alpha_7\,\mathscr{E}_i\,\mathscr{E}_j) - \eta\,[(\beta_2+\alpha_3)\,\mathscr{E}_m\,n_m\,\delta_{ij} - (\alpha_3\,\mathscr{E}_j\,n_i + \beta_2\,\mathscr{E}_i\,n_j)] + \\
&+ \eta^2[\beta_1(n_i\,n_j - \delta_{ij}) + \beta_7\,\varepsilon_{jkl}\,\varepsilon_{ipq}\,\mathscr{E}_q\,\mathscr{E}_k\,n_p\,n_l]\,.
\end{aligned} \tag{10.2}$$

a) Longitudinal electric field

If the electric field $\mathscr{E}$ is parallel to the direction of propagation of the wave, then we take this common direction as the x_3-axis of the coordinate system x. Introducing

$$n_i = \delta_{3i} \quad \text{and} \quad \mathscr{E}_i = \mathscr{E}\,\delta_{3i} \tag{10.3}$$

into (10.2), we obtain

$$\chi_{ij} = (\alpha_1\,\delta_{ij} + \alpha_7\,\mathscr{E}^2\,\delta_{3i}\,\delta_{3j}) - \eta\,\mathscr{E}(\beta_2+\alpha_3)\,(\delta_{ij} - \delta_{3i}\,\delta_{3j}) - \eta^2\beta_1(\delta_{ij} - \delta_{3i}\,\delta_{3j})\,. \tag{10.4}$$

Whence,

$$\|\chi_{ij}\| = \begin{Vmatrix}
\alpha_1 - (\beta_2+\alpha_3)\,\mathscr{E}\,\eta - \beta_1\eta^2, & 0, & 0\\
0, & \alpha_1 - (\beta_2+\alpha_3)\,\mathscr{E}\,\eta - \beta_1\eta^2, & 0\\
0, & 0, & \alpha_1 + \alpha_7\,\mathscr{E}^2
\end{Vmatrix}. \tag{10.5}$$

Discounting the possibility that $\alpha_1+\alpha_7\mathscr{E}^2=0$, the secular equation (7.8) then yields

$$\eta=\frac{-\mathscr{E}(\beta_2+\alpha_3)\pm[(\beta_2+\alpha_3)^2\,\mathscr{E}^2+4\,\alpha_1\,\beta_1]^{\frac{1}{2}}}{2\beta_1}. \tag{10.6}$$

Unless $\beta_2+\alpha_3=0$, these two solutions for η correspond to waves travelling with different complex velocities parallel to the x_3-axis. We may expect that one of these waves will travel in the positive direction of the x_3-axis and the other in the negative direction.

Introducing (10.5) into (7.6), we obtain $e_3=0$. Consequently, for each of the waves the relation (8.13) is satisfied and the wave is accordingly transverse. Also, it is easily seen that the solutions (10.6) for η give the inverse complex velocities for waves linearly-polarized in an arbitrary direction normal to the direction of propagation.

We note, as was pointed out in §7, that if $\eta^+>0$ and $\eta^-<0$, or if $\eta^+<0$ and $\eta^->0$, the system is unstable. Consequently, α_1, β_1, β_2 and α_3 must be such that this situation does not arise for either wave. A number of conclusions regarding the values of α_1, β_1, β_2 and α_3 might be drawn from this fact by purely algebraic reasoning. For example, the situation cannot exist in which $[(\beta_2+\alpha_3)^2\mathscr{E}^2+4\alpha_1\beta_1]^{\frac{1}{2}}/\beta_1$ is real and $\mathscr{E}(\beta_2+\alpha_3)/\beta_1$ is purely imaginary.

b) Transverse electric field

We may, without loss of generality, choose the reference system with the x_3-axis parallel to the direction of wave propagation and the x_2-axis parallel to $\mathscr{E}$. We then write

$$n_i=\delta_{3i} \quad \text{and} \quad \mathscr{E}_i=\mathscr{E}\,\delta_{2i}. \tag{10.7}$$

Introducing these relations into (10.2), we obtain

$$\begin{aligned}\chi_{ij}=(\alpha_1\,\delta_{ij}+\alpha_7\,\mathscr{E}^2\,\delta_{2i}\,\delta_{2j})+\eta\,\mathscr{E}\,(\alpha_3\,\delta_{3i}\,\delta_{2j}+\beta_2\,\delta_{2i}\,\delta_{3j})+{}\\ +\eta^2[\beta_1(\delta_{3i}\,\delta_{3j}-\delta_{ij})+\beta_7\,\mathscr{E}^2(\delta_{2i}\,\delta_{2j}+\delta_{3i}\,\delta_{3j}-\delta_{ij})].\end{aligned} \tag{10.8}$$

Thus,

$$\|\chi_{ij}\|=\left\|\begin{matrix}\alpha_1-\eta^2(\beta_1+\beta_7\,\mathscr{E}^2), & 0, & 0\\ 0, & \alpha_1+\alpha_7\,\mathscr{E}^2-\beta_1\,\eta^2, & \beta_2\,\mathscr{E}\,\eta\\ 0, & \alpha_3\,\mathscr{E}\,\eta, & \alpha_1\end{matrix}\right\|. \tag{10.9}$$

The solutions for η^2 of the secular equation (7.8) are therefore

$$\eta^2=\frac{\alpha_1}{\beta_1+\beta_7\,\mathscr{E}^2}, \tag{10.10}$$

or

$$\eta^2=\frac{\alpha_1\,(\alpha_1+\alpha_7\,\mathscr{E}^2)}{\alpha_1\,\beta_1+\alpha_3\,\beta_2\,\mathscr{E}^2}. \tag{10.11}$$

The solutions (10.10) for η correspond to waves travelling with equal speeds in the positive and negative directions of the x_3-axis. Unless

$$\alpha_7\,\beta_7\,\mathscr{E}^2=\alpha_3\,\beta_2-\beta_1\,\alpha_7-\alpha_1\,\beta_7 \tag{10.12}$$

$e_2=e_3=0$ for each of these waves and consequently they are transverse waves, linearly polarized in the x_1-direction.

The solutions (10.11) for η also correspond to waves travelling with equal speeds in the positive and negative directions of the x_3-axis. For these waves, unless the relation (10.12) is satisfied, it is seen from (7.6) and (10.9) that

$$e_1 = 0$$

and

$$e_3/e_2 = -\alpha_3 \mathscr{E} \eta/\alpha_1. \tag{10.13}$$

The waves are consequently skew and elliptically polarized in the $x_2 x_3$-plane. From (10.13), we see that $\boldsymbol{e}$ has the form

$$\boldsymbol{e} = \xi(0, 1, \zeta), \tag{10.14}$$

where

$$\zeta = -\alpha_3 \mathscr{E} \eta/\alpha_1 \tag{10.15}$$

and ξ is an arbitrary complex number. Following the discussion in §9, we again see that the semi-axes L_1 and L_2 of the ellipse executed by the vector $\boldsymbol{E}^+$ are given by (9.18) and the values of the parameter φ determining their direction are given by (9.19), where ζ is now given by (10.15).

11. Appendix

We suppose that the magnetic intensity and electric displacement fields, $\boldsymbol{H}^*$ and $\boldsymbol{D}^*$, at time t are functionals of the electric and magnetic induction fields, $\boldsymbol{E}^*(\tau)$ and $\boldsymbol{B}^*(\tau)$, over all times preceding the instant t. Thus,

$$\boldsymbol{D}^* = \boldsymbol{\Phi} \underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}}$$

and

$$\boldsymbol{H}^* = \boldsymbol{\Psi} \underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}}. \tag{11.1}$$

We assume that the functionals $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ satisfy the relations

$$\boldsymbol{\Phi} \underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau-h), \boldsymbol{B}^*(\tau-h)\right]}} = \boldsymbol{\Phi} \underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}}$$

and

$$\boldsymbol{\Psi} \underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau-h), \boldsymbol{B}^*(\tau-h)\right]}} = \boldsymbol{\Psi} \underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}}. \tag{11.2}$$

The material considered then possesses invariable heredity.

We also assume that each of the functionals $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ may be particularly dependent on the values of the arguments and of their time derivatives of various orders at time $\tau = t$ and is continuous for $-\infty < \tau < t$. Using the notation

$$\begin{aligned} \boldsymbol{E}^*(t) &= \boldsymbol{E}^* = \boldsymbol{E}^{*(0)}, \quad & \boldsymbol{E}^{*(\mu)} &= [d^\mu \boldsymbol{E}^*(\tau)/d\tau^\mu]_{\tau=t}, \\ \boldsymbol{B}^*(t) &= \boldsymbol{B}^* = \boldsymbol{B}^{*(0)}, \quad & \boldsymbol{B}^{*(\mu)} &= [d^\mu \boldsymbol{B}^*(\tau)/d\tau^\mu]_{\tau=t}, \\ & (\mu = 1, 2, \ldots) \end{aligned} \tag{11.3}$$

we suppose that the functionals $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ may be expressed in the forms

$$\begin{aligned} \boldsymbol{\Phi}\underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}} &= \sum \boldsymbol{f} \cdot \boldsymbol{F}\underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}}, \\ \boldsymbol{\Psi}\underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}} &= \sum \boldsymbol{g} \cdot \boldsymbol{G}\underset{\tau=-\infty}{\overset{t}{\left[\boldsymbol{E}^*(\tau), \boldsymbol{B}^*(\tau)\right]}}, \end{aligned} \tag{11.4}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are differentiable functions of $\boldsymbol{E}^{*(\mu)}$ and $\boldsymbol{B}^{*(\mu)}$ ($\mu = 0, 1, 2, \ldots$) and $\boldsymbol{F}$ and $\boldsymbol{G}$ are continuous functionals of their arguments.

We now assume that the fields $\boldsymbol{E}^*(\tau)$ and $\boldsymbol{B}^*(\tau)$ are the resultants of static fields $\mathcal{E}$ and $\mathcal{B}$ and small varying fields $\varepsilon\boldsymbol{E}(\tau)$ and $\varepsilon\boldsymbol{B}(\tau)$, where ε is a small constant, as given by equations (2.1). We have, neglecting terms of higher degree than the first in ε,

$$\begin{gathered}
\boldsymbol{f}(\boldsymbol{E}^{*(\mu)},\boldsymbol{B}^{*(\mu)})=\boldsymbol{f}(\boldsymbol{E}^{*(\mu)},\boldsymbol{B}^{*(\mu)})\big|_{\varepsilon=0}+\varepsilon\sum_{\mu=0}^{\mu}\left\{\frac{\partial f}{\partial E^{*(\mu)}}\bigg|_{\varepsilon=0}\boldsymbol{E}^{(\mu)}+\frac{\partial f}{\partial B^{*(\mu)}}\bigg|_{\varepsilon=0}\boldsymbol{B}^{(\mu)}\right\}\\
\text{and}\\
\mathop{\boldsymbol{F}}_{\tau=-\infty}^{t}[\boldsymbol{E}^*(\tau),\boldsymbol{B}^*(\tau)]=\boldsymbol{F}(\mathcal{E},\mathcal{B})+\varepsilon\int_{-\infty}^{t}\boldsymbol{\varphi}(t-\tau)\,\boldsymbol{E}(\tau)\,d\tau+\varepsilon\int_{-\infty}^{t}\boldsymbol{\psi}(t-\tau)\,\boldsymbol{B}(\tau)\,d\tau,
\end{gathered}\tag{11.5}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are functions of $\mathcal{E}$ and $\mathcal{B}$. Here, we use the notation $\boldsymbol{E}^{(0)}=\boldsymbol{E}$, $\boldsymbol{B}^{(0)}=\boldsymbol{B}$, $\boldsymbol{E}^{(\mu)}=[d^\mu\boldsymbol{E}(\tau)/d\tau^\mu]_{\tau=t}$, $\boldsymbol{B}^{(\mu)}=[d^\mu\boldsymbol{B}(\tau)/d\tau^\mu]_{\tau=t}$. We also use the notation

$$\boldsymbol{F}(\mathcal{E},\mathcal{B})=\mathop{\boldsymbol{F}}_{\tau=-\infty}^{t}[\boldsymbol{E}^*(\tau),\boldsymbol{B}^*(\tau)]\big|_{\varepsilon=0},\tag{11.6}$$

observing that $\mathcal{E}(=\boldsymbol{E}^*(\tau)|_{\varepsilon=0})$ and $\mathcal{B}(=\boldsymbol{B}^*(\tau)|_{\varepsilon=0})$ are time-independent fields. The second of the results (11.5) follows directly from the integral representation of a continuous functional. We note that $\partial\boldsymbol{f}/\partial\boldsymbol{E}^{*(\mu)}|_{\varepsilon=0}$ and $\partial\boldsymbol{f}/\partial\boldsymbol{B}^{*(\mu)}|_{\varepsilon=0}$ are functions of $\mathcal{E}$ and $\mathcal{B}$.

It follows from (11.4) and (11.5) that, neglecting terms of higher degree than the first in ε, the functional $\boldsymbol{\Phi}[\boldsymbol{E}^*(\tau),\ \boldsymbol{B}^*(\tau)]$ may be expressed in the form

$$\begin{aligned}
\mathop{\boldsymbol{\Phi}}_{\tau=-\infty}^{t}[\boldsymbol{E}^*(\tau),\boldsymbol{B}^*(\tau)]=\Phi(\mathcal{E},\mathcal{B})+\varepsilon\sum_{\mu=0}^{\mu}(\boldsymbol{\alpha}_\mu\boldsymbol{E}^{(\mu)}+\boldsymbol{\beta}_\mu\boldsymbol{B}^{(\mu)})+\int_{-\infty}^{t}\boldsymbol{\varphi}(t-\tau)\,\boldsymbol{E}(\tau)\,d\tau+{}\\
+\int_{-\infty}^{t}\boldsymbol{\psi}(t-\tau)\,\boldsymbol{B}(\tau)\,d\tau,
\end{aligned}\tag{11.7}$$

where the $\boldsymbol{\alpha}$'s, $\boldsymbol{\beta}$'s, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are functions of $\mathcal{E}$ and $\mathcal{B}$ which are, in general, different from those in (11.5).

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